Journal of Statistical Physics, Vol. 39, Nos. 1/2, 1985

Random Walks on Lattices with Points of Two Colors. II. Some Rigorous Inequalities for Symmetric Random Walks

W. Th. F. den Hollander[1] and P. W. Kasteleyn[1]

Received September 24, 1984; revised November 12, 1984

We continue our investigation of a model of random walks on lattices with two kinds of points, "black" and "white." The colors of the points are stochastic variables with a translation-invariant, but otherwise arbitrary, joint probability distribution. The steps of the random walk are independent of the colors. We are interested in the stochastic properties of the sequence of consecutive colors encountered by the walker. In this paper we first summarize and extend our earlier general results. Then, under the restriction that the random walk be symmetric, we derive a set of rigorous inequalities for the average length of the subwalk from the starting point to a first black point and of the subwalks between black points visited in succession. A remarkable difference in behavior is found between subwalks following an odd-numbered and subwalks following an even-numbered visit to a black point. The results can be applied to a trapping problem by identifying the black points with imperfect traps.

KEY WORDS: Random walks; inhomogeneous lattice; colored points; average length of successive runs; ergodic theorems; perfect and imperfect traps.

1. INTRODUCTION

In a previous paper[1] we have introduced a model of a random walk on a lattice of which the points can carry two different colors, "black" and "white." The colors are (not necessarily independent) frozen-in stochastic variables. We have obtained rigorous results for a number of stochastic properties of the sequence of consecutive colors encountered by the walker while stepping through the lattice. The model may serve to describe certain transport processes in disordered media, such as the diffusion and trapping

[1] Instituut-Lorentz voor Theoretische Natuurkunde, Nieuwsteeg 18, 2311 SB, Leiden, The Netherlands.

0022-4715/85/0400-0015$04.50/0

of "particles" in a medium with static traps. Our aim is to obtain results which are valid for a broad range of different types of diffusion and disorder, and therefore we have kept the model as general as possible. In addition, this model is an example of a doubly stochastic process which we feel is interesting in its own right.

The definitions are as follows. Consider an infinite d-dimensional lattice L and suppose that the points of L are colored *black* and *white* according to a given joint probability distribution $\mathscr{P}$. $\mathscr{P}$ is assumed to be *translation invariant.* The probability that a given point $l \in L$ is black is thereby independent of l; we denote it by q and assume that $q > 0$. Next consider a random walk on L, starting at the origin and proceeding according to a given probability distribution p for single steps. It is assumed that p is *independent* of the coloring of L (and translation invariant as usual).

A color distribution $\mathscr{P}$ may be defined by attributing probabilities to *local* configurations of black and white points, i.e., partitions of *finite* subsets of L into a set of black points and a set of white points with the colors of points outside the subset unspecified. If this is done in a consistent way, these probabilities determine a unique probability distribution for the infinite lattice (as is guaranteed by the so-called extension theorem; see Ref. 2, Vol. 2, p. 118). Two local configurations that are obtained from each other by a translation must be assigned equal probability in order to acquire translation invariance. $\mathscr{P}$ is otherwise completely arbitrary. Examples are (see Ref. 1): (i) the *random* distribution; (ii) (translation-invariant) *periodic* distributions; (iii) the *uniform* distribution; (iv) (translation-invariant) *grand canonical* distributions (Gibbs states).

The step distribution p may be chosen to be any function $p: L \to \mathbb{R}$ with $p(l) \geqslant 0$ and $\sum_{l \in L} p(l) = 1$, where $p(l)$ is the probability of a step over the lattice vector l; p assigns probabilities to the individual steps of the walker, independent of the colors by assumption and, of course, independent of previous steps.

We are interested in the walker's visits to black points, called "hits," more in particular in the stochastic properties of the number of steps made before the first hit and between successive hits. A subwalk between successive hits we call a *run* and, for convenience, we call the subwalk to the first hit the zeroth run. In Ref. 1 we have considered the following two stochastic processes:

(0) The process $(n_0, n_1, n_2, \ldots)$, where n_i is the length of the ith run, $i \geqslant 0$ $(n_0 \geqslant 0;\ n_i \geqslant 1,\ i \geqslant 1)$.

(1) The process $(0, n_1, n_2, \ldots)$ obtained from (0) by the restriction that n_0 be zero, i.e., that the origin be black.

Both processes are entirely determined by the two independent probability distributions $\mathscr{P}$ and p. The number of black points visited may be finite, i.e., it may happen that only a finite number of runs is completed. Averages such as $\langle n_i \rangle$ will, however, be understood as conditional averages given that the ith run in *completed* (i.e., given that at least $i+1$ black points are visited). Furthermore, averages in process 1 are in turn *conditional* averages in process 0 given that $n_0 = 0$. Nevertheless, we prefer to speak of 0 and 1 as separate processes; accordingly we denote averages by $\langle \cdots \rangle_0$ and $\langle \cdots \rangle_1$, respectively.

In Ref. 1 we have shown, using a simple "renewal-type" argument (of the type used, e.g., in Ref. 2, Vol. 1, Chap. 13) in combination with our assumption of translation invariance, that the stochastic properties of either process can be expressed rigorously in terms of those of the other. From this relationship we have derived several general properties. Here is a list of the main results:

A. In process 0 the probability that a first black point is hit at step n_0 is a *monotone* nonincreasing function of n_0.

B. Let f_i, $i \geqslant 0$, be the probability that in process 0 at least the ith run is completed. Let p_i, $i \geqslant 1$, be the corresponding probability in process 1. Then

$$f_i = f_0, \qquad \text{for all } i \geqslant 1 \tag{1.1}$$

$$p_i = 1, \qquad \text{for all } i \geqslant 1 \tag{1.2}$$

C. Process 1 is *stationary;* hence the stochastic variables n_i are identically distributed in this process. Furthermore, each of the moments of n_1 in process 1 can be linearly expressed in terms of lower moments of n_0 in process 0. In particular,

$$\langle n_1 \rangle_1 = f_0 q^{-1} \tag{1.3}$$

$$\langle n_1^2 \rangle_1 = f_0 q^{-1}(1 + 2\langle n_0 \rangle_0) \tag{1.4}$$

D. In process 0 the moments of n_i with $i \geqslant 1$ can be expressed in terms of *correlations* in process 1. In particular,

$$\langle n_i \rangle_0 = \langle n_1 n_{i+1} \rangle_1 / \langle n_1 \rangle_1, \qquad i \geqslant 1 \tag{1.5}$$

E. In process 0 the following inequalities hold:

$$\langle n_0 \rangle_0 \geqslant \tfrac{1}{2}(f_0 q^{-1} - 1) \tag{1.6}$$

$$\langle n_0 \rangle_0 \geqslant \tfrac{1}{2}(\langle n_i \rangle_0 - 1), \qquad \text{for all} \quad i \geqslant 1 \tag{1.7}$$

In process 0 the n_i with $i \geqslant 1$ are in general *not* identically distributed. Furthermore, both in process 0 and in process 1 the lengths of the successive runs are in general *correlated.* This stems from the fact that different black points may have different "environments," which is the reason why it is difficult to study the processes in detail. Process 0 is physically the more interesting one and has our main interest. Process 1 serves more or less as an "auxiliary" process.

The above remarks summarize the results of Ref. 1. The outline of this paper is as follows. In Section 2 we study the probability f_0 in more detail. The result is:

F. In process 0

$$f_0 = 1 - \mathscr{P}[\bar{L} \text{ is white}] \tag{1.8}$$

where $\bar{L}$ is the smallest sublattice of L to which the random walk is confined.

The results obtained thus far are valid for arbitrary L, $\mathscr{P}$, and p and are therefore necessarily of a modest nature. Stronger results can be obtained as more specific assumptions are introduced. In Section 3, which constitutes the main part of the paper, we focus on one such specific assumption, viz. that the random walk is *symmetric*, i.e., that $p(l) = p(-l)$ for all l. Under this assumption we derive the following set of inequalities, assuming (without loss of generality) that (a) the color distribution is *extremal* (i.e., $\mathscr{P}$ cannot be decomposed into two distinct translation-invariant components), (b) the random walk covers the *whole* lattice (i.e., $\bar{L} = L$), so that $f_0 = 1$:

G. For the zeroth run in process 0

$$\langle n_0 \rangle_0 \geqslant (1-q)^2/q(1-X) \tag{1.9}$$

where $X := \operatorname{Prob}[n_1 = 1 \mid n_0 = 0] = q^{-1} \sum_{l \in L} p(l)\, \mathscr{P}[0 \text{ and } l \text{ are black}]$.

H. For the higher runs in process 0 let $\Delta_i := \langle n_i \rangle_0 - q^{-1}$. Then

$$\begin{gathered} \Delta_1 \geqslant \Delta_3 \geqslant \Delta_5 \geqslant \Delta_7 \geqslant \cdots \geqslant 0 \\ \Delta_1 \geqslant |\Delta_2|,\ \Delta_3 \geqslant |\Delta_4|,\ \Delta_5 \geqslant |\Delta_6|, \ldots \end{gathered} \tag{1.10}$$

We further show:

I. For periodic color distributions $\Delta_i \to 0$ as $i \to \infty$, exponentially fast, irrespective of the random walk. For general color distributions decay is expected to occur in most cases, but it may be slower than exponential.

In Section 3.1 we first consider periodic color distributions. The unit cell of the periodic pattern will be completely arbitrary. The arguments used are essentially probabilistic and are based on simple matrix algebra. The derivation presented for this case will set the stage for the extension to arbitrary (translation-invariant) color distributions, which is given in Section 3.2 and requires the use of certain ergodic theorems. In Section 4 we apply our results by identifying the black points with imperfect traps. Section 5 is devoted to a discussion, including some examples and a few references to related results in the literature.

The reader who is not interested in the derivation of F–I may wish to skip Sections 2 and 3 and go straight to Section 4.

When in the following we speak of runs we shall mean runs in process 0, unless stated otherwise. As in the previous paper, the assumption of translation invariance will play a key role in the calculations.

2. THE PROBABILITY f_0

The probability f_0 that the zeroth run is completed plays an important part in Ref. 1 and appears in many of the formulas. Of course, this probability may depend on L, $\mathscr{P}$, and p. In this section we shall study f_0 in more detail.

Let us use the symbol $P_n(l)$ to denote the probability that the walker visits point $l \in L$ at step n. Let further

$$L^+ := \{l \in L\colon P_n(l) > 0 \text{ for some } n \geqslant 0\}$$

L^+ is the set of all points that can be reached by the walker in a finite number of steps; it depends on p. We shall first prove that

$$f_0 = 1 - \mathscr{P}[L^+ \text{ is white}] \tag{2.1}$$

Proof. When L^+ is white there are no black points that the walker can reach and the zeroth run cannot be completed ($n_0 = \infty$). To prove Eq. (2.1) we must show that $n_0 < \infty$ with probability 1 when it is given that L^+ contains a black point.

The plan of the proof is to use Eq. (1.2). Now Eq. (1.2) states that given that the origin is black the walker will with probability 1 hit arbitrarily many black points, or in other words, there exists with probability 1 an infinite sequence of steps at which the walker hits a black point. By the translation invariance of $\mathscr{P}$ it immediately follows that also the following is true: given that l is black there exists with probability 1 an infinite sequence of step numbers $k_0(=0) < k_1 < k_2 < \dots$ such that $l_{k_j} + l$ is black, where l_n stands for the point visited at step n. These step numbers are, of course, stochastic variables. It is important that the above statement is true for *all* $l \in L$.

Next, let $l \in L^+$ and let m be the smallest integer with $P_m(l) > 0$. Assume that l is black and let $m_0(=0) < m_1 < m_2 < \ldots$ be the infinite sequence of the smallest elements of $(k_j)_{j \geqslant 0}$ with the property that $m_{j+1} - m_j \geqslant m$ for all j. We define the following sequence of events: E_j, $j \geqslant 0$, is the event $\{m_i < \infty$ for $i \leqslant j$ and $l_{m_j+m} = l_{m_j} + l\}$, or in other words, the event that the first $j+1$ step numbers in the sequence $(m_j)_{j \geqslant 0}$ indeed exist and that the visit to l_{m_j} is followed by a visit to the black point $l_{m_j} + l$ at the mth subsequent step. Clearly, the event E_j has probability $\rho_j = P_m(l) > 0$, independent of j. Moreover, since $m_{j+1} - m_j \geqslant m$ the events are independent. It follows from the second Borel–Cantelli lemma[2] that, as $\sum_j \rho_j = \infty$, with probability 1 arbitrarily many among the events E_j will occur.

We have now reached the following result: given that some point of L^+ is black the walker will with probability 1 hit arbitrarily many black points. But, trivially, this implies that $f_i = \mathscr{P}[L^+$ contains a black point$]$ for all $i \geqslant 0$. Equation (2.1) is the special case for $i = 0$ (note that from Eq. (1.1) we already knew that all the f_i are equal). This completes the proof. ∎

Equation (2.1) may be slightly strengthened. Let

$$\bar{L} := \{l \in L: l = l' - l'' \text{ for some } l', l'' \in L^+\}$$

$\bar{L}$ is the smallest sublattice of L that contains L^+ (see Ref. 3, p. 15); it is the lattice on which the random walk "actually takes place." In many cases $\bar{L} = L$, but not in all. We shall now prove that

$$f_0 = 1 - \mathscr{P}[\bar{L} \text{ is white}] \tag{2.2}$$

Proof. Since $L^+ \subset \bar{L}$ it follows that $\mathscr{P}[\bar{L}$ is white$] \leqslant \mathscr{P}[L^+$ is white$]$. To prove Eq. (2.2) we must show that the equality sign holds. To do so we shall again use the translation invariance of $\mathscr{P}$.

The proof will depend on the following remarkable property. Let $L' := \{l_k\}$, $k \in \mathbb{Z}$, be *any* line of points in L, i.e., $l_k = l_a + k l_b$ for some $l_a, l_b \in L$ with $l_b \neq 0$. If L' contains *one* black point then with probability 1 it contains *infinitely many*, extending in both directions. Indeed, let $c_k := \mathscr{P}[l_k$ is black, $l_{k'}$ is white for all $k' > k]$. Then we have $\mathscr{P}[L'$ contains a black point and a white positive half-line$] = \sum_{k \in \mathbb{Z}} c_k$. Now obviously $\sum_{k \in \mathbb{Z}} c_k \leqslant 1$. By the translation invariance of $\mathscr{P}$, however, c_k is independent of k and so it must be that $c_k = 0$. Hence $\sum_{k \in \mathbb{Z}} c_k = 0$, which proves that if

L' contains one black point it contains with probability 1 infinitely many on the positive half-line $k \geqslant 0$. For the negative half-line the argument, of course, runs the same. This proves the statement.

We shall use the above-mentioned property in combination with some elementary properties of the structure of L^+. When $L^+ = \bar{L}$ there is nothing to prove as then Eqs. (2.1) and (2.2) are identical. For the rest of the proof we shall therefore assume that $L^+ \neq \bar{L}$. We want to show that $\mathscr{P}[L^+$ is white$] - \mathscr{P}[\bar{L}$ is white$] = \mathscr{P}[L^+$ is white, $\bar{L} \backslash L^+$ contains a black point$] = 0$.

Let us begin by specializing to the case $d = 1$. We may assume that $\bar{L} = L(= \mathbb{Z})$, since it will be clear that there is no loss of generality in doing so. (Note that the *degenerate* random walk with $p(0) = 1$ has $L^+ = \bar{L} = \{0\}$ and is therefore excluded by our assumption that $L^+ \neq \bar{L}$.) Let $\Sigma := \{l \in L\colon p(l) > 0\}$. L^+ is the set of all finite sums of elements of Σ (i.e., L^+ is the additive semigroup generated by Σ). Suppose first that $p(l) = 0$ for $l < 0$. Then all elements of L^+ are nonnegative. Because L^+ is not contained in any proper sublattice of $\mathbb{Z}$, the greatest common divisor of the elements of Σ is 1. This implies that $l \in L^+$ for l sufficiently large, so that L^+ contains a positive half-line. It now follows that $\mathscr{P}[L^+$ is white, $L \backslash L^+$ contains a black point$] \leqslant \mathscr{P}[\mathbb{Z}$ contains a black point and a white positive half-line$] = 0$, which is the required equality. When $p(l) = 0$ for $l > 0$ the result is the same. When, finally, Σ has both positive and negative elements it is readily seen that $L^+ = \bar{L}$, which is the trivial case excluded. This proves Eq. (2.2) for $d = 1$.

The proof for $d \geqslant 2$ follows almost as a corollary. L^+ may have a variety of forms depending on L and p, but since L and $\mathscr{P}$ are completely arbitrary there is again no loss of generality in assuming that $\bar{L} = L$. To get the desired result it now suffices to observe that for any point $l \in L$ there exists a line through l that has a half-line in common with L^+ (this derives, in fact, from a simple group-theoretical property). It follows that given that l is black there are with probability 1 infinitely many black points in L^+. Since l is arbitrary this again implies that $\mathscr{P}[L^+$ is white, $L \backslash L^+$ contains a black point$] = 0$, which completes the proof of Eq. (2.2). ∎

Equation (2.2) is a strong result: once it is known that there is *some* black point in $\bar{L}$ it follows that with probability 1 *some* black point is hit. For recurrent random walks this is not surprising, for in this case always $L^+ = \bar{L}$ and *each* point of L^+ is hit with probability 1 (see Ref. 3, p. 19), but for transient random walks it is. It should be emphasized, however, that the generality of Eq. (2.2) is due entirely to the translation invariance of $\mathscr{P}$. The proof shows that $\bar{L}$ is either white or contains infinitely many black points extending in *all* directions. This property explains some of the background of Eqs. (1.2) and (2.2). In Section 3.2 we shall take a closer look at the effect of the translation invariance on the coloring of L and show that with probability 1 an "asymptotic density" of black points exists.

From Eq. (2.2) it follows that in almost all cases of physical interest $f_0 = 1$. Indeed, when the random walk is *aperiodic* (in the sense of Spitzer, Ref. 3, p. 20) we have $\bar{L} = L$ (by definition), in which case $f_0 = 1$ if the possibility that L is white has zero probability. For the random distribution $f_0 = 1$ if only we exclude the degenerate random walk which has $f_0 = q$ (in all other cases $|\bar{L}| = \infty$). It will be clear that when $\bar{L} \neq L$ the problem is in a sense "ill posed" and, instead of $\mathscr{P}$, one may then as well consider the restriction of $\mathscr{P}$ to $\bar{L}$, which is obviously translation invariant on $\bar{L}$.

Finally, Eq. (2.2) shows that cases with $f_0 < 1$ are in a sense nothing but trivial extensions of cases with $f_0 = 1$. Assume $\bar{L} = L$. If $f_0 < 1$ there is a positive probability that L is white, but then it is always possible to reduce the problem by writing $\mathscr{P}$ as the (unique) *convex linear combination* of two translation-invariant color distributions $\mathscr{P}'$ and $\mathscr{P}''$, viz. $\mathscr{P} = f_0 \mathscr{P}' + (1 - f_0)\mathscr{P}''$, with $\mathscr{P}'[L \text{ is white}] = 0$ and $\mathscr{P}''[L \text{ is white}] = 1$. Since we are interested in completed runs only, $\mathscr{P}''$ is the "irrelevant" part that does not contribute to the averages in our model. $\mathscr{P}'$, on the other hand, covers all relevant events and thus one may reduce the problem by "scaling" $\mathscr{P}$ to $\mathscr{P}'$. This also explains why the probabilities q and f_0 always appear in the combination f_0/q: for $\mathscr{P}'$ we have the corresponding probabilities $q' = q/f_0$ and $f_0' = f_0/f_0 = 1$, and q' is the "effective" probability that a point is black once the problem is reduced by scaling.

In the following we shall assume, for reasons which will become clear later, that $\mathscr{P}[L \text{ is black}] = 0$. By the same argument it will be clear that this minor restriction involves no loss of generality.

3. RIGOROUS INEQUALITIES FOR SYMMETRIC RANDOM WALKS

We shall henceforth center interest on the *moments* $\langle n_i \rangle_0$, $i \geqslant 0$. Whereas the probabilities f_i equal 1 for practically all choices of L, $\mathscr{P}$, and p, these moments depend strongly on this choice.

As observed in Ref. 1, one may derive from Eqs. (1.3)–(1.5), noting that $\langle n_1^2 \rangle_1 \geqslant \langle n_1 \rangle_1^2$ and $\langle n_1 n_{i+1} \rangle_1 \leqslant \frac{1}{2}\langle n_1^2 + n_{i+1}^2 \rangle_1 = \langle n_1^2 \rangle_1$, the following two inequalities mentioned in the Introduction:

$$\langle n_0 \rangle_0 \geqslant \tfrac{1}{2}(f_0 q^{-1} - 1) \tag{3.1}$$

$$\langle n_0 \rangle_0 \geqslant \tfrac{1}{2}(\langle n_i \rangle_0 - 1), \qquad \text{for all } i \geqslant 1 \tag{3.2}$$

These equations reduce to equalities for a few special cases.

With a simple scaling argument Eq. (3.1) may be slightly refined to obtain a stronger bound that depends on p:

$$\langle n_0 \rangle_0 \geqslant \tfrac{1}{2}(f_0 q^{-1} - 1)/[1 - p(0)] \tag{3.1a}$$

Unfortunately, however, it does not seem easy to do better in general. No upper bound was found in Ref. 1. Furthermore, for the runs with number $i \geqslant 1$ no lower bound was found. Here it should be noted that for *any given* $q > 0$ [and $p(0)$ fixed] one easily constructs simple (though somewhat extreme) examples where $\langle n_0 \rangle_0$ can take *arbitrarily large* values or $\langle n_i \rangle_0$ can take *any* value > 1 for *given* $i \geqslant 1$. Thus Eqs. (3.1a) and (3.2) are by nature weak. One may expect stronger results if one places restrictions on $\mathscr{P}$ or on p, or both.

In the following we shall investigate the case where p is *symmetric*. We shall derive a set of rigorous inequalities which are valid for arbitrary (translation-invariant) color distributions. In Section 3.1 we first consider periodic color distributions. The extension to arbitrary color distributions is given in Section 3.2.

3.1. Periodic Color Distributions

We begin with some definitions. A periodic color *configuration* is an arrangement of black and white points in L such that L can be divided into identical *finite* unit cells ("blocks") having identical (local) color configurations (in other words, there is a periodic pattern of colors). A *translation-invariant* periodic color *distribution* is obtained by choosing an arbitrary periodic color configuration and assigning equal probability to all distinct configurations obtained from the chosen one by a translation. Examples are: (i) a *strictly* periodic distribution, where the black points form a sublattice of L and the (smallest) unit cell contains one black point; (ii) a *pair*-periodic distribution, where the (smallest) unit cell contains two black points.

Consider first an *arbitrary* random walk p on L. We shall find it convenient to change our point of view in two ways. First, since we are not interested in the positions at which the walker hits the black points we shall consider the random walk as taking place on a single unit cell with *periodic boundary conditions* imposed. This unit cell we denote by $\tilde{L}$. Second, rather than sticking to our description with a fixed starting point for the walker and a translation-invariant color distribution, we shall fix the positions of the colors and, instead, allow the walker to start with equal probability at any point of $\tilde{L}$. The two descriptions are obviously equivalent; however, the latter description facilitates the discussion somewhat.

Let, then, N be the number of points in $\tilde{L}$, $B = \{l_1, ..., l_t\} \subset \tilde{L}$ the set of black points in $\tilde{L}(1 \leqslant t < N)$ and $W = \tilde{L} \backslash B$ the set of white points. $\tilde{L}$ and the sets B and W are *completely arbitrary*. The results which we derive depend only on the existence of a unit cell. We may assume that $t < N$, i.e., that W is not empty, as otherwise the model is trivial. Of course, $q = t/N$.

(I) The Zeroth Run. The zeroth run plays a special role in that it may start either on a black or on a white point. If the walker starts in B then $n_0 = 0$; if he starts in W then he can go through a succession of visits to points of W before hitting a point of B. We define

$$p_{ll'} := \text{probability of a step from } l \text{ to } l';\ l, l' \in \tilde{L}$$

(taking into account the periodic boundary conditions!). Further, let $\boldsymbol{p} := (p_{ll'})_{l,l' \in W}$ denote the $(N-t) \times (N-t)$ matrix that has as elements the stepping probabilities between the *white* points. Of course, $\boldsymbol{p}$ depends on p as well as on the shape and the coloring of $\tilde{L}$.

Now the average $\langle n_0 \rangle_0$ can be expressed in terms of $\boldsymbol{p}$ as follows. Let

$$w_n := \text{probability that after } n \text{ steps the walker has not yet hit a black point; } n \geqslant 0.$$

In order not to hit B the walker must start in W and make steps between points of W only. Recalling that with probability N^{-1} the walker may start at any point of $\tilde{L}$, we therefore have

$$w_n = N^{-1} \sum_{l,l' \in W} (\boldsymbol{p}^n)_{ll'}, \qquad n \geqslant 0 \tag{3.3}$$

We shall assume that $f_0 = 1$. (This condition will be removed later.) Then $w_n \to 0$ as $n \to \infty$ and by the monotonicity of w_n

$$\langle n_0 \rangle_0 = \sum_{n=1}^{\infty} n(w_{n-1} - w_n) = \sum_{n=0}^{\infty} w_n \tag{3.4}$$

(see Ref. 2, Vol. 1, p. 265). Hence by Eq. (3.3)

$$\langle n_0 \rangle_0 = N^{-1} \sum_{l,l' \in W} (\boldsymbol{1} + \boldsymbol{p} + \boldsymbol{p}^2 + \cdots)_{ll'} \tag{3.5}$$

where $\boldsymbol{1}$ denotes the $(N-t) \times (N-t)$ unit matrix. Note that $\boldsymbol{p}$ does *not* include any steps from W to B that are needed by the walker to reach a black point. These steps will appear in the calculation at a later stage.

Obviously, $\sum_{l' \in W} p_{ll'} \leqslant 1$ for all $l \in W$. This property is expressed by saying that the matrix $\boldsymbol{p}$ is *substochastic*. The condition $f_0 = 1$ implies that strict inequality holds for *at least one* $l \in W$ (as otherwise the walker could never escape from W once he had started in W), and thus $\boldsymbol{p}$ is *strictly substochastic*. Now it is well known that the eigenvalues of a (nonnegative) strictly substochastic matrix are all strictly smaller in modulus than unity (see, e.g., Refs. 4 and 5). Since we shall need this property later we give the proof.

Proof. First assume that $\boldsymbol{p}$ is irreducible. Then the well-known Perron–Frobenius theorem for nonnegative (square) matrices[4,5] states that $\boldsymbol{p}$ has a real eigenvalue $\lambda_1 > 0$ with the following properties:

(i) λ_1 is nondegenerate and with it are associated strictly positive left and right eigenvectors.

(ii) $|\lambda| \leqslant \lambda_1$ for any other eigenvalue λ of $\boldsymbol{p}$.

(iii) $\lambda_1 \leqslant \max_{l \in W}(\sum_{l' \in W} p_{ll'})$ and $\lambda_1 \leqslant \max_{l' \in W}(\sum_{l \in W} p_{ll'})$.

By (iii) one has $\lambda_1 \leqslant 1$. However, $\lambda_1 = 1$ is excluded, since if $\boldsymbol{x} := (x_l)_{l \in W}$ is the left eigenvector associated with λ_1 then $\lambda_1 = 1$ would imply that

$$\sum_{l' \in W} x_{l'} = \sum_{l \in W} x_l \left(\sum_{l' \in W} p_{ll'} \right) \leqslant \sum_{l \in W} x_l$$

and hence, by (i), that $\sum_{l' \in W} p_{ll'} = 1$ for all $l \in W$. With (ii) this completes the proof for irreducible $\boldsymbol{p}$. When $\boldsymbol{p}$ is reducible one can through a permutation of its rows and columns obtain a matrix of the form

$$\begin{bmatrix} \boldsymbol{p}_{11} & \boldsymbol{0} \\ \boldsymbol{p}_{21} & \boldsymbol{p}_{22} \end{bmatrix}$$

where $\boldsymbol{p}_{11}$ and $\boldsymbol{p}_{22}$ are square matrices and $\boldsymbol{p}_{11}$ is either zero or irreducible. A repetition of the above argument shows that $\boldsymbol{p}_{11}$ cannot have an eigenvalue 1, and the proof is completed by induction. ∎

Thus it is seen that the condition $f_0 = 1$ entails that $|\lambda| < 1$ for all eigenvalues λ of $\boldsymbol{p}$. This in turn implies that the inverse of $\boldsymbol{1} - \boldsymbol{p}$ exists, so that we may write for Eq. (3.5)

$$\langle n_0 \rangle_0 = N^{-1}(\boldsymbol{e}, (\boldsymbol{1} - \boldsymbol{p})^{-1} \boldsymbol{e}) \tag{3.6}$$

where $\boldsymbol{e}$ denotes the $(N - t)$-vector with all elements equal to 1 and $(.,.)$ stands for the vector inner product. An important consequence of the existence of $(\boldsymbol{1} - \boldsymbol{p})^{-1}$ is that $\langle n_0 \rangle_0 < \infty$.

So far we have not yet made any assumption concerning the random walk p. Even though Eq. (3.6) may not be a very suitable starting point for a detailed calculation of $\langle n_0 \rangle_0$, it will serve us here to obtain a bound in the case of a symmetric random walk. Therefore we now, as promised, assume that p is *symmetric*, i.e.,

$$p(l) = p(-l), \qquad \text{for all } l \in L \tag{3.7}$$

Equation (3.7) implies that the matrix $\boldsymbol{p}$ is symmetric (because a step and its reverse between any two points of $\tilde{L}$ are equally probable by symmetry). Therefore all eigenvalues of $\boldsymbol{p}$ are real and it immediately follows that $\mathbf{1} - \boldsymbol{p}$ is *positive definite*. This is the crucial property in the argument: we can now use a matrix inequality known as the Kantorovich inequality (see Ref. 6, p. 117 and Ref. 7, p. 69), which gives in our case

$$(\boldsymbol{e}, (\mathbf{1}-\boldsymbol{p})\, \boldsymbol{e})(\boldsymbol{e}, (\mathbf{1}-\boldsymbol{p})^{-1}\, \boldsymbol{e}) \geqslant (\boldsymbol{e}, \boldsymbol{e})^2 \tag{3.8}$$

and yields with Eq. (3.6)

$$\langle n_0 \rangle_0 \geqslant (\boldsymbol{e}, \boldsymbol{e})^2 / N(\boldsymbol{e}, (\mathbf{1}-\boldsymbol{p})\, \boldsymbol{e}) \tag{3.9}$$

The right-hand side of Eq. (3.9) is easy to evaluate. Indeed, we have $(\boldsymbol{e}, \boldsymbol{e}) = N - t$ and $(\boldsymbol{e}, \boldsymbol{pe}) = \sum_{l,l' \in W} p_{ll'} = N - 2t + \sum_{l,l' \in B} p_{ll'}$, where now the probabilities of steps to and from *black* points appear as we use that $\sum_{l \in \tilde{L}} p_{ll'} = 1$ for all $l' \in \tilde{L}$ and $\Sigma_{l' \in \tilde{L}} p_{ll'} = 1$ for all $l \in \tilde{L}$; the latter equalities follow from the condition $\sum_{l \in L} p(l) = 1$. Thus we finally arrive at

$$\langle n_0 \rangle_0 \geqslant (1-q)^2 / q(1-X) \tag{3.10}$$

with $q = t/N$ and $X := t^{-1} \sum_{l,l' \in B} p_{ll'}$.

The condition $f_0 = 1$ is easily removed. As observed at the end of Section 2, cases with $f_0 < 1$ are trivial extensions of cases with $f_0 = 1$, and a simple scaling argument has made it clear that in the general case q should be replaced by q/f_0. In our setting $\tilde{L}$ may be partitioned into two sets S' and S'' from which the walker has probability 1 and 0, respectively, to reach B. S' is a sublattice of $\tilde{L}$ (or a union of sublattices), $B \subset S'$ and $f_0 = |S'|/N$. Because $\langle n_0 \rangle_0$ is a conditional average given that the zeroth run is completed, only those walks that start from S' will contribute. S' takes over the role of $\tilde{L}$, $f_0' = 1$, and $q' = |B|/|S'| = q/f_0$. Hence the general result is

$$\langle n_0 \rangle_0 \geqslant (f_0 q^{-1} - 1)^2 / f_0 q^{-1} (1-X) \tag{3.11}$$

Equation (3.11) is the first of a series of inequalities that are the object of this section. Note that the connection between process 0 and process 1, as seen in Eq. (1.4), reappears through X: X is the probability that $n_1 = 1$ given that $n_0 = 0$. Note further that $X < 1$. X depends on p as well as on the shape and the coloring of $\tilde{L}$. In all cases, however, $X \geqslant p(0)$ so that we have the weaker but simpler bound

$$\langle n_0 \rangle_0 \geqslant (f_0 q^{-1} - 1)^2 / f_0 q^{-1} [1 - p(0)] \tag{3.12}$$

which is to be compared with Eq. (3.1a). Here only the variable q appears as prominent and the dependence on $\mathscr{P}$ and p has disappeared nearly altogether.

Equation (3.11) is stronger than Eq. (3.1a). When $f_0 = 1$ this is obvious for $q \leqslant \frac{1}{2}$. For $q > \frac{1}{2}$, on the other hand, note that $X \geqslant 1 + (1 - q^{-1})[1 - p(0)]$ and hence $\langle n_0 \rangle_0 \geqslant (1 - q)/[1 - p(0)]$. Incidentally, the latter inequality is trivial.

For several special cases Eq. (3.11) reduces to an equality, e.g., for any unit cell and a random walk that is "indifferent" with respect to $\tilde{L}$, in the sense that from any point of $\tilde{L}$ the probability of a jump to any other point is $1/(N-1)$ [here $f_0 = 1$ and $p(0) = 0$]. An example is the case with $L = \mathbb{Z}$, the strictly periodic distribution with $q = 1/3$ and the simple random walk (where steps to nearest-neighbor points have equal probability and other steps are not allowed). In the derivation of Eq. (3.11) we have used of $\boldsymbol{p}$ only its symmetry. Obviously, the translation invariance of p places *additional restrictions* on $\boldsymbol{p}$ (to be more specific, $\boldsymbol{p}$ is obtained from a cyclic stochastic matrix by deleting certain rows and columns). If this fact were exploited Eq. (3.11) could perhaps be strengthened further.

(II) The Runs with Number $i \geqslant 1$. The runs with number $i \geqslant 1$ all start from a black point. This is, however, all that they have in common. As pointed out earlier, the n_i are in general *not* independent *nor* are they identically distributed, to the effect that the moments $\langle n_i \rangle_0$ for different i take different values.

To study $\langle n_i \rangle_0$, $i \geqslant 1$, we shall make explicit use of the relations between process 0 and process 1 established in Ref. 1. We define

$$\Delta_i := \langle n_i \rangle_0 - f_0 q^{-1}, \qquad i \geqslant 1 \tag{3.13}$$

$$\gamma_i := \langle n_1 n_{i+1} \rangle_1 - \langle n_1 \rangle_1 \langle n_{i+1} \rangle_1, \qquad i \geqslant 1 \tag{3.14}$$

With Eqs. (1.3) and (1.5) we have

$$\Delta_i = \gamma_i / f_0 q^{-1} \tag{3.15}$$

This relation says that the amount by which $\langle n_i \rangle_0$ differs from $f_0 q^{-1}$ is directly related to the *correlation* of the runs 1 and $i+1$ in process 1. By studying γ_i we shall be able to get information about Δ_i. In particular, since $f_0 q^{-1} > 0$ either Δ_i and γ_i have the same sign or they are both zero.

Process 1 is in a sense easier than process 0 because it is stationary. Also in process 1, however, the n_i are in general *not* independent to the effect that the γ_i, and hence the Δ_i, are $\neq 0$. Note that in Eq. (3.13) the term $f_0 q^{-1}$ is, by Eq. (1.3), the average of n_i in process 1 (!). Thus one could say that in process 0 the zeroth run, through its mere existence, has an effect on all subsequent runs. Note that $\langle n_i \rangle_0 < \infty$ for all $i \geqslant 1$ by Eq. (3.2), as $\langle n_0 \rangle_0 < \infty$.

(a) The First Run. Let us consider the first run to begin with. Again we first assume that p is arbitrary and begin with some definitions:

$T_n(i \to j)$:= probability for the walker, when starting from $l_i \in B$, to make a run of exactly n steps to $l_j \in B$; $n \geqslant 1$; $i, j = 1, \dots, t$.

$p_{n_1 n_2}$:= probability that in process 1 the first run has length n_1 and the second run length n_2. (3.16)

With these definitions we shall write out γ_1.

In process 1 the walker may start with probability t^{-1} at any point of B and so

$$p_{n_1 n_2} = t^{-1} \sum_{i,j,k} T_{n_1}(i \to j)\, T_{n_2}(j \to k) \tag{3.17}$$

From Eq. (3.16) we get

$$\begin{aligned} \langle n_1 n_2 \rangle_1 &= \sum_{n_1, n_2} n_1 n_2\, p_{n_1 n_2} \Big/ \sum_{n_1, n_2} p_{n_1 n_2} \\ &= t^{-1} \sum_{i,j,k} S_{ij} S_{jk} \Big/ t^{-1} \sum_{i,j,k} T_{ij} T_{jk} \end{aligned} \tag{3.18}$$

where we introduce

$$T_{ij} := \sum_n T_n(i \to j) \tag{3.19a}$$

$$S_{ij} := \sum_n n T_n(i \to j) \tag{3.19b}$$

The probabilities T_{ij}, $i, j = 1, \dots, t$, form a matrix of what may be called "transition" probabilities between different "states": T_{ij} is the total probability of a run from l_i to l_j. Because $\tilde{L}$ is finite

$$\sum_j T_{ij} = 1, \qquad \text{for all } i \tag{3.20a}$$

[See also Eq. (1.2).] Also the following is true:

$$\sum_i T_{ij} = 1, \qquad \text{for all } j \tag{3.20b}$$

This is seen by comparing the random walk p with the *reversed* random walk $\tilde{p}$ obtained from p by defining $\tilde{p}(l) = p(-l)$, $l \in L$. For each $n \geqslant 1$ the probabilities $T_n(i \to j)[p]$ and $T_n(i \to j)[\tilde{p}]$ in Eq. (3.16), corresponding to p and $\tilde{p}$, are related as

$$T_n(i \to j)[\tilde{p}] = T_n(j \to i)[p]$$

This gives $T_{ij}[\tilde{p}] = T_{ji}[p]$, so that Eq. (3.20b) follows from Eq. (3.20a), which is valid for all p. In all cases therefore $\boldsymbol{T} := (T_{ij})$ is what is called a *doubly stochastic* matrix.

Using Eq. (3.20a) we may simplify Eq. (3.18) a little bit to

$$\langle n_1 n_2 \rangle_1 = t^{-1} \sum_{i,j,k} S_{ij} S_{jk} \tag{3.21}$$

The product $\langle n_1 \rangle_1 \langle n_2 \rangle_1$, which is the second term of γ_1 in Eq. (3.14), is known from Eq. (1.3), but we shall want for it an expression similar to Eq. (3.21). Following a similar line of reasoning as above we get

$$\langle n_1 \rangle_1 = t^{-1} \sum_{i,j} S_{ij} \tag{3.22a}$$

$$\langle n_2 \rangle_1 = t^{-1} \sum_{i,j,k} T_{ij} S_{jk} = t^{-1} \sum_{j,k} S_{jk} \tag{3.22b}$$

where we use Eq. (3.20b). Combining with Eq. (3.21) we thus arrive at

$$\gamma_1 = t^{-1} \sum_{i,j,k} S_{ij} S_{jk} - \left\{ t^{-1} \sum_{i,j} S_{ij} \right\}^2 \tag{3.23}$$

Now we are ready to use the *symmetry* of p, which will again be seen to be of crucial importance. From Eq. (3.7) it follows that $p = \tilde{p}$ and hence that $\boldsymbol{T}$ and $\boldsymbol{S} := (S_{ij})$ are symmetric. Defining

$$S_i := \sum_j S_{ij}, \qquad i = 1, \ldots, t \tag{3.24}$$

and noting that by the symmetry also $S_j = \sum_i S_{ij}$, we may then write

$$\gamma_1 = t^{-1} \sum_i S_i^2 - \left\{ t^{-1} \sum_i S_i \right\}^2 \tag{3.25}$$

Now the right-hand side of this equation has the pleasant property that it can be written as a sum of squares (!):

$$\gamma_1 = t^{-2} \sum_{i<j} (S_i - S_j)^2 \tag{3.26}$$

and it thus immediately follows that $\gamma_1 \geqslant 0$ and, by Eq. (3.15), $\Delta_1 \geqslant 0$ or with Eq. (3.13)

$$\langle n_1 \rangle_0 \geqslant f_0 q^{-1} \tag{3.27}$$

This is the desired inequality for the first run.

The equality sign in Eq. (3.27) holds *if and only if* the S_i in Eq. (3.24) are *all equal.* S_i has a simple interpretation: it is *the average length of a run starting from black point labeled i.* When $t=1$ the sum in Eq. (3.26) is empty and the equality sign holds. This is the case of a *strictly* periodic distribution. [Note that it follows from Eq. (3.23) that for this particular case $\langle n_1 \rangle_0 = f_0 q^{-1}$ also for asymmetric random walks; here the equality for general p is a well-known result found by Montroll[8].] When $t=2$ the sum in Eq. (3.26) is not empty; in this case, however, we have not only $T_n(1 \to 2) = T_n(2 \to 1)$ by the symmetry of the random walk but also $T_n(1 \to 1) = T_n(2 \to 2)$ by the inversion symmetry of the unit cell, so that both $S_{12} = S_{21}$ and $S_{11} = S_{22}$ hence $S_1 = S_2$ and again $\langle n_1 \rangle_0 = f_0 q^{-1}$. This is the case of a *pair*-periodic distribution. When $t \geqslant 3$ we do in general *not* have equality (unless of course the arrangement of black points is strictly or pair-periodic on a smaller scale); equality then holds only for special choices of p, $\tilde{L}$, and B.

(b) The Runs 2,3,.... Further inequalities follow from simple matrix algebra. Equations (3.13)–(3.15) serve as our starting point.

With arguments similar to those presented above it is found that Eq. (3.23) generalizes to

$$\gamma_k = t^{-1} \sum_{i,j,m,n} S_{ij} (\boldsymbol{T}^{k-1})_{jm} S_{mn} - \left\{ t^{-1} \sum_{i,j} S_{ij} \right\}^2, \qquad k \geqslant 1 \tag{3.28}$$

Here the power $\boldsymbol{T}^{k-1}$ "bridges the gap" between the first and the $(k+1)$st run in the first term of Eq. (3.14). When we use the *symmetry* of the random walk this equation simplifies further to

$$\gamma_k = t^{-1} (\boldsymbol{s}, (\boldsymbol{T}^{k-1} - t^{-1} \boldsymbol{E})\, \boldsymbol{s}) \tag{3.29}$$

where $\boldsymbol{s}$ is the t vector with components S_i, $i = 1, \dots, t$, as given by Eq. (3.24), $\boldsymbol{E}$ is the $t \times t$ matrix with all elements equal to 1 and $(.,.)$ again denotes the vector inner product.

To proceed we make the following observations.

(i) $\boldsymbol{TE} = \boldsymbol{ET} = \boldsymbol{E}$ by Eqs. (3.20a, b), so $\boldsymbol{T}^{k-1}$ and $t^{-1}\boldsymbol{E}$ have a common base of eigenvectors. Hence each eigenvalue of $\boldsymbol{T}^{k-1} - t^{-1}\boldsymbol{E}$ is the difference of the corresponding eigenvalues of $\boldsymbol{T}^{k-1}$ and $t^{-1}\boldsymbol{E}$.

(ii) Both $\boldsymbol{T}$ and $\boldsymbol{E}$ are symmetric and have real eigenvalues. By the Perron–Frobenius theorem the eigenvalues of $\boldsymbol{T}$ fall in the interval $[-1, 1]$. The largest eigenvalue is 1 (which may be degenerate if $\boldsymbol{T}$ is reducible) and one eigenvalue 1 corresponds to the eigenvector (1,..., 1). $\boldsymbol{E}$ has an eigenvalue t corresponding to the same eigenvector and a $(t-1)$-fold degenerate eigenvalue 0. Thus, when $1 = \tau_1 \geqslant \tau_2 \geqslant \cdots \geqslant \tau_t \geqslant -1$ are the eigenvalues of $\boldsymbol{T}$, the eigenvalues of $\boldsymbol{T}^{k-1} - t^{-1}\boldsymbol{E}$ are 0 and the powers $\tau_2^{k-1},..., \tau_t^{k-1}$.

(iii) Because $\boldsymbol{T}$ and $\boldsymbol{E}$ are symmetric there exists a matrix $\boldsymbol{O}$, which is orthogonal, such that $\boldsymbol{O}(\boldsymbol{T}^{k-1} - t^{-1}\boldsymbol{E})\,\boldsymbol{O}^{-1} = \boldsymbol{D}^{k-1}$, with the diagonal matrix

$$\boldsymbol{D} = \begin{bmatrix} 0 & & \cdots & 0 \\ & \tau_2 & & \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & \tau_t \end{bmatrix}$$

When we now define $\boldsymbol{u} = \boldsymbol{Os}$ we get from Eq. (3.29)

$$\gamma_k = t^{-1}(\boldsymbol{u}, \boldsymbol{D}^{k-1}\boldsymbol{u}) = t^{-1} \sum_{i=2}^{t} U_i^2 \tau_i^{k-1}, \qquad k \geqslant 1 \tag{3.30}$$
$$1 \geqslant \tau_2 \geqslant \cdots \geqslant \tau_t \geqslant -1$$

where the U_i are the components of $\boldsymbol{u}$.

From Eq. (3.30) a number of interesting inequalities can be deduced. We use Eq. (3.15), and read off for k *odd*:

(1) $\Delta_k \geqslant 0$; for $k \geqslant 3$ the Δ_k are either all $=0$ or all >0;

(2) $\Delta_k \geqslant \Delta_m$ for all $m > k$ (m both odd and even);

(3) $\Delta_k + \Delta_{k+1} \geqslant 0$ and hence with (2): $\Delta_k \geqslant |\Delta_{k+1}|$;

(4) When $\tau_2 \neq 1$ and $\tau_t \neq -1$ then $\Delta_k \to 0$ exponentially fast with k as $k \to \infty$, including the even-numbered runs.

Thus we have as a general result the set of inequalities

$$\Delta_1 \geqslant \Delta_3 \geqslant \Delta_5 \geqslant \Delta_7 \geqslant \cdots \geqslant 0,$$
$$\Delta_1 \geqslant |\Delta_2|, \Delta_3 \geqslant |\Delta_4|, \Delta_5 \geqslant |\Delta_6|, ... \tag{3.31}$$

Whereas the Δ_k for *k odd* display a "smooth" *monotonic* decay, surprisingly enough the *even*-numbered runs show *no such general behavior*. In fact, examples show that there is a *rich variety* (!) in behavior for Δ_{2k}, depending on the choice of the colors in the unit cell and the step distribution of the random walk. In some cases all Δ_{2k} are positive, in others all are negative, but there exist cases where different signs occur. As an example for the latter situation, take for $\tilde{L}$ a ring of six lattice points (the ring structure takes care of the periodic boundary conditions) with three black points, two of which are neighbors and the third at a nonneighboring position, and take for the random walk one with steps of probability 3/7 over one lattice spacing and 1/14 over two. A straightforward calculation shows that for this particular example $\Delta_2 > \Delta_4 > 0$, whereas $\Delta_6 < \Delta_8 < \cdots < 0$.

In many cases one has the smooth behavior

$$\Delta_1 \geqslant \Delta_2 \geqslant \Delta_3 \geqslant \Delta_4 \geqslant \Delta_5 \geqslant \cdots \geqslant 0 \tag{3.31a}$$

From Eq. (3.30) it is clear that this will certainly occur in all cases where the eigenvalues of $\boldsymbol{T}$ are all $\geqslant 0$. From the so-called Gerchgorin theorem[9] together with Eqs. (3.20a, b) it follows that the eigenvalues fall in the set

$$\bigcup_{i=1}^{l} \{\tau \in [-1, 1]: |\tau - T_{ii}| \leqslant 1 - T_{ii}\} = [2(\min_i T_{ii}) - 1, 1]$$

For Eq. (3.31a) to hold it therefore suffices that $T_{ii} \geqslant \frac{1}{2}$ for all i. This is so, for instance, in the following two cases: (i) $p(0) \geqslant \frac{1}{2}$, irrespective of *all* other details; (ii) the simple random walk on any ring of points with *any* color arrangement that does *not* have two black points as neighbors.

In some cases where Eq. (3.31a) does not hold one encounters an *oscillating* decay of the type

$$\Delta_1 \geqslant -\Delta_2 \geqslant \Delta_3 \geqslant -\Delta_4 \geqslant \Delta_5 \geqslant \cdots \geqslant 0 \tag{3.31b}$$

This may appear, for instance, when there is some underlying symmetry in the arrangement of the colors and some of the U_i in Eq. (3.30) are zero. As an example consider a simple random walk on a ring of six points with three black points next to each other. One finds after a short calculation: $\Delta_k = (-\frac{1}{2})^{k+1}$, $k \geqslant 1$.

To classify the different types of behavior for the even-numbered runs one would need more detailed information about $\boldsymbol{T}$ and $\boldsymbol{S}$. It turns out that this presents a very complicated problem, since in general little is known about these matrices in detail. Although Eqs. (3.31a, b) appear only as special cases of Eq. (3.31), examples tend to show that the monotonic decay and, to a lesser extent, the oscillating decay are predominant. In most cases the decay becomes asymptotically either monotonic or oscillating as $k \to \infty$.

A special situation occurs when $\tau_2 = 1$ (and $U_2 \neq 0$): in that case the Δ_k do *not* decay to zero. This, however, is possible only when $\boldsymbol{T}$ is reducible, since when $\boldsymbol{T}$ is irreducible the eigenvalue 1 is always nondegenerate (see Ref. 5, p. 120). Irreducibility of $\boldsymbol{T}$ means that from each black point the walker can eventually reach all other black points (i.e., there are no disjoint sets of black points between which the walker cannot "cross over"). It will be clear that in the reducible case the problem is in a sense "ill posed" and can always be reduced to the irreducible case. In the latter case also the eigenvalue -1 is nondegenerate. From the Gerchgorin theorem one sees that $\tau_t = -1$ can occur only when $T_{ii} = 0$ for *some i*. By the irreducibility it can occur only when $T_{ii} = 0$ for *all i* (see Ref. 5, p. 121). Together with the symmetry of the random walk the latter condition is so strong that it is fulfilled only in the trivial case where the walker cannot make any step between a black and a white point.

Finally, in Eq. (3.31) the equality signs hold in a few special cases, notably for the strictly and the pair-periodic distribution. (In the latter case $U_2 = 0$ due to the symmetry.) A remarkable situation occurs when $\Delta_1 > 0$ and $\Delta_k = 0$ for $k \geqslant 2$. This happens when $\boldsymbol{T}$ has an eigenvalue zero and all other eigenvalues carry a coefficient zero. This may be illustrated by the following example: a ring of five points with three black points, two of which are neighbors and the third at the remaining nonneighboring position, and a random walk with steps of probability $[1 + (13)^{1/2}]/12$ over one lattice spacing and $[5 - (13)^{1/2}]/12$ over two. In this example $\boldsymbol{T}$ is not invertible (!) and, so to say, *projects* the Δ_k, $k \geqslant 2$, onto zero, in the sense that for any given $k \geqslant 2$, but *not* for $k = 1$, the three black points each have probability $\frac{1}{3}$ to be hit as the kth black point, which brings the result back to Eq. (1.3).

3.2. Extension to Arbitrary Color Distributions

So far we have only considered periodic color distributions. The fact that the unit cell of the periodicity was completely arbitrary makes one suspect that the results of the previous section may be generalized. As we shall see, this is indeed the case and the extension can be made to arbitrary (translation-invariant) color distributions. It turns out, however, that the extension is far from trivial. In fact, the approach followed in Section 3.1 will serve only as a guide and on our way we shall encounter some new and interesting problems that have to be dealt with. Thus the extension is more than just a piece of formalism.

In the general case there is no unit cell and we return to our original description with a fixed starting point for the walker (at the origin) and a translation-invariant color distribution. For simplicity we shall from now on assume that $L = \mathbb{Z}^d$; the arguments are easily generalized to arbitrary lattices.

(I) The Zeroth Run. Our aim is to generalize Eq. (3.10). Let $\mathscr{B} := \{B\colon B \subset L\}$ be the set of all subsets of L. A color configuration in L will henceforth be *identified* with the set that consists of all the black points and $\mathscr{B}$ may therefore represent the set of all color configurations. For a given B let $W := L \backslash B$ and let $\boldsymbol{p}^{(B)} := (p(l' - l))_{l,l' \in W}$ be the matrix of stepping probabilities between the white points. The set W is either finite or (countably) infinite. By the translation invariance, W is empty with probability 1 when it is finite (see Section 2). Since, however, we have assumed that $\mathscr{P}[L \text{ is black}] = \mathscr{P}[W \text{ is empty}] = 0$, it follows that with probability 1 the matrix $\boldsymbol{p}^{(B)}$ is (countably) infinite.

Let $I[E] \in \{0, 1\}$ denote the indicator stochastic variable of the event E. A little reflection shows that instead of Eq. (3.5) we now have

$$\langle n_0 \rangle_0 = \overline{I[0 \in W] \sum_{l \in W} (\mathbf{1} + \boldsymbol{p}^{(B)} + \boldsymbol{p}^{(B)2} + \cdots)_{0l}}^{\mathscr{B}} \tag{3.32}$$

where the bar denotes the average over $\mathscr{B}$ with respect to $\mathscr{P}$ and $\mathbf{1}$ is the (countably) infinite unit matrix. Here we must again assume that $f_0 = 1$. Because of the translation invariance we may choose instead of 0 any point of L as the starting point for the walker and we are therefore free to write

$$\langle n_0 \rangle_0 = \overline{I[l \in W] \sum_{l' \in W} (\mathbf{1} + \boldsymbol{p}^{(B)} + \boldsymbol{p}^{(B)2} + \cdots)_{ll'}}^{\mathscr{B}} \qquad \text{for any } l \in L \tag{3.33}$$

The powers of $\boldsymbol{p}^{(B)}$ are well-defined (see, e.g., Ref. 4, p. 161). It is not at all clear, however, whether or not the inverse of $\mathbf{1} - \boldsymbol{p}^{(B)}$ exists with probability 1 and thus whether or not $\langle n_0 \rangle_0 < \infty$. This is a problem not encountered in the periodic case. We shall return to this point in the discussion.

We shall arrive at our result by a *truncation* method together with a suitable limit procedure. Let

$$L_n := \{l \in L\colon |l^i| \leqslant n,\ i = 1,..., d\}, \qquad n \geqslant 0$$

and let $\boldsymbol{p}_n^{(B)}$ be the truncation of $\boldsymbol{p}^{(B)}$ obtained by deleting all rows and columns that correspond to white points *outside* L_n, i.e., $\boldsymbol{p}_n^{(B)}$ is the matrix of stepping probabilities between the white points that fall in L_n. Clearly we have

$$\langle n_0\rangle_0 \geqslant |L_n|^{-1} \overline{\sum_{l,l' \in L_n} I[l \in W]\, I[l' \in W](\mathbf{1}_n^{(B)} + \boldsymbol{p}_n^{(B)} + \boldsymbol{p}_n^{(B)2} + \cdots)_{ll'}}^{\mathscr{B}} \qquad \text{for any } n \tag{3.34}$$

where $\mathbf{1}_n^{(B)}$ is the unit matrix of the same order as $\boldsymbol{p}_n^{(B)}$, i.e., of order $|L_n \cap W|$. To get Eq. (3.34) one first averages in Eq. (3.33) over l in L_n and then truncates $\boldsymbol{p}^{(B)}$ (observe that the elements of $\boldsymbol{p}^{(B)}$ are nonnegative). The advantage of Eq. (3.34) over Eq. (3.33) is not only that l and l' appear symmetrically but also that $\boldsymbol{p}_n^{(B)}$ is a *finite* matrix. Working with Eq. (3.34) we shall avoid some difficulties that are connected with infinite matrices (see, e.g., Ref. 4, Chap. 6) and that would, at least for our purpose, unnecessarily complicate the calculations.

As we shall see in a moment, the right-hand side of Eq. (3.34) is finite for all n. It is also monotone nondecreasing in n. We shall derive for this right-hand side an inequality valid for all n and then take the limit $n \to \infty$ to obtain the desired inequality for $\langle n_0\rangle_0$. For finite n there may be a positive probability that $L_n \cap W$ is empty, in which case $\boldsymbol{p}_n^{(B)}$ is not defined. However, since the sequence $(L_n)_{n \geqslant 0}$ is monotone and $\lim_{n\to\infty} L_n = L$ it follows that $\lim_{n\to\infty} \mathscr{P}[L_n \text{ is black}] = \mathscr{P}[L \text{ is black}]$, and as the latter probability is zero by assumption the probability that $L_n \cap W$ is empty tends to zero as $n \to \infty$.

If we exclude the degenerate random walk ($p(0) = 1$), then because L_n is finite there is for all $B \in \mathscr{B}$ a step of positive probability that will bring the walker from a point inside $L_n \cap W$ (if not empty) to a point outside. Therefore $\boldsymbol{p}_n^{(B)}$ is for all B and n strictly substochastic so that the inverse of $\mathbf{1}_n^{(B)} - \boldsymbol{p}_n^{(B)}$ exists. Thus we may write for Eq. (3.34)

$$\begin{aligned}\langle n_0\rangle_0 &\geqslant |L_n|^{-1} \overline{\sum_{l,l' \in L_n \cap W} (\mathbf{1}_n^{(B)} - \boldsymbol{p}_n^{(B)})^{-1}_{ll'}}^{\mathscr{B}} \\ &= |L_n|^{-1} \overline{(\boldsymbol{e}_n^{(B)}, (\mathbf{1}_n^{(B)} - \boldsymbol{p}_n^{(B)})^{-1} \boldsymbol{e}_n^{(B)})}^{\mathscr{B}} \qquad \text{for any } n\end{aligned} \tag{3.35}$$

where $\boldsymbol{e}_n^{(B)}$ is the vector of order $|L_n \cap W|$ with all elements equal to 1. Note that the right-hand side of Eq. (3.35) is finite for all n.

Now we are ready to use the *symmetry* of the random walk. This property implies that for all B and n the matrix $\boldsymbol{p}_n^{(B)}$ is symmetric, so that $\mathbf{1}_n^{(B)} - \boldsymbol{p}_n^{(B)}$ is positive definite. The Kantorovich inequality gives

$$\langle n_0\rangle_0 \geqslant |L_n|^{-1} \overline{(\boldsymbol{e}_n^{(B)}, \boldsymbol{e}_n^{(B)})^2/(\boldsymbol{e}_n^{(B)}, (\mathbf{1}_n^{(B)} - \boldsymbol{p}_n^{(B)})\, \boldsymbol{e}_n^{(B)})}^{\mathscr{B}} \qquad \text{for all } n \tag{3.36}$$

We proceed as follows. Let

$$q_n^{(B)} := |L_n \cap B| / |L_n| \tag{3.37a}$$

denote the fraction of black points in L_n for given B. We have $(e_n^{(B)}, e_n^{(B)}) = |L_n \cap W|$ and

$$\begin{aligned}(e_n^{(B)}, p_n^{(B)} e_n^{(B)}) &= \sum_{l,l' \in L_n \cap W} p(l-l') \\ &= |L_n \cap W| - |L_n \cap B| + \sum_{l,l' \in L_n \cap B} p(l-l') + R_n^{(B)}\end{aligned}$$

where

$$R_n^{(B)} := \sum_{l \notin L_n} \sum_{l' \in L_n \cap B} p(l-l') - \sum_{l \in L_n \cap W} \sum_{l' \notin L_n} p(l-l') \tag{3.37b}$$

plays the role of a rest term. Defining

$$X_n^{(B)} := |L_n \cap B|^{-1} \sum_{l,l' \in L_n \cap B} p(l-l') \tag{3.37c}$$

we thus get

$$\langle n_0 \rangle_0 \geqslant \overline{(1-q_n^{(B)})^2 / (q_n^{(B)}[1-X_n^{(B)}] - |L_n|^{-1} R_n^{(B)})}^{\mathscr{B}} \qquad \text{for all } n \tag{3.38}$$

Now we consider the limit $n \to \infty$ of Eq. (3.38). First we show that

$$\lim_{n \to \infty} |L_n|^{-1} R_n^{(B)} = 0, \qquad \text{for all } B \in \mathscr{B} \tag{3.39a}$$

Proof. By the symmetry of the random walk it follows from Eq. (3.37b) that $|R_n^{(B)}| \leqslant \sum_{l \in L_n} \sum_{l' \notin L_n} p(l-l')$ for all B. The latter sum does not depend on B. Let $c_m := \sum_{l \in L, |l| \geqslant m} p(l)$, $m \geqslant 0$, and $L_{m,n} := L_n \backslash L_{n-m}$, $m < n$ (the "shell" of thickness m between the cubes L_n and L_{n-m}). Then one writes

$$\begin{aligned}\sum_{l \in L_n} \sum_{l' \notin L_n} p(l-l') &= \sum_{l \in L_{n-m}} \sum_{l' \notin L_n} p(l-l') + \sum_{l \in L_{m,n}} \sum_{l' \notin L_n} p(l-l') \\ &\leqslant |L_{n-m}| \, c_m + |L_{m,n}|\end{aligned}$$

and for m fixed this gives

$$\lim_{n \to \infty} |L_n|^{-1} |R_n^{(B)}| \leqslant \lim_{n \to \infty} |L_n|^{-1} \{ |L_{n-m}| \, c_m + |L_{m,n}| \} = c_m$$

Noting that $c_m \to 0$ as $m \to \infty$ one sees that Eq. (3.39a) follows. ∎

Having thus disposed of the rest term we are next faced with the question whether or not also the limits

$$q^{(B)} := \lim_{n \to \infty} q_n^{(B)} \tag{3.39b}$$

and

$$X^{(B)} := \lim_{n \to \infty} X_n^{(B)} \tag{3.39c}$$

exist. This is not immediately obvious. Indeed, it is easy to construct color configurations for which these limits do *not* exist. However, the translation invariance entails, as we shall show in a moment, that $q^{(B)}$ and $X^{(B)}$ *exist with probability 1*. This is weaker but enough for our purpose, and we can now safely take the limit in Eq. (3.38) to obtain

$$\langle n_0 \rangle_0 \geqslant \overline{(1-q^{(B)})^2/q^{(B)}[1-X^{(B)}]}^{\mathscr{B}} \tag{3.40}$$

thus completing the generalization of Eq. (3.10).

The existence with probability 1 of the limits in Eqs. (3.39b, c) follows from an *ergodic theorem* for so-called (super)additive stochastic processes[10–12]. Consider first Eq. (3.39b). For any finite set $S \subset L$ let $N_S^{(B)} := |S \cap B|$ denote the number of black points of B that fall in S. The random variable $N_S^{(B)}$ has the following three properties:

(i) By the translation invariance the probability distributions of $N_S^{(B)}$ and $N_{S+l}^{(B)}$ are identical for all $l \in L$, where $S+l$ is the set obtained from S after a translation over l ("stationarity").

(ii) For any two disjoint sets S and S': $N_{S \cup S'}^{(B)} = N_S^{(B)} + N_{S'}^{(B)}$, for all $B \in \mathscr{B}$ ("additivity").

(iii) $0 \leqslant \overline{N_S^{(B)}}^{\mathscr{B}} \leqslant |S|$, for all $B \in \mathscr{B}$ ("integrability").

From a theorem by Pitt[10] (which is a generalization to higher dimensions of the well-known ergodic theorem of Birkhoff) it then follows that $\lim_{n \to \infty} |S_n|^{-1} N_{S_n}^{(B)}$ exists with probability 1 for any sequence of finite sets $(S_n)_{n \geqslant 0}$ such that $\lim_{n \to \infty} |S_n| = \infty$, provided this sequence satisfies the following *regularity* conditions: (1) $S_n \subset L_n$ for all n; (2) $|S_n|/|L_n|$ is bounded from below; (3) S_n is convex in L for all n. Furthermore, under these conditions the limit does *not* depend on the sequence chosen. Obviously, if we choose $S_n = L_n$ the regularity is guaranteed and this proves the existence with probability 1 of $q^{(B)}$. In general $q^{(B)}$ will be a stochastic variable.

Next consider Eq. (3.39c). For any finite set $S \subset L$ we now define $M_S^{(B)} := \sum_{l,l' \in S \cap B} p(l-l')$. This random variable has properties similar to those of $N_S^{(B)}$, except that (ii) is replaced by the following:

(ii)′ For any two disjoint sets S and S': $M^{(B)}_{S \cup S'} \geqslant M^{(B)}_S + M^{(B)}_{S'}$, for all $B \in \mathscr{B}$ ("superadditivity").

The existence with probability 1 of $\lim_{n \to \infty} |S_n|^{-1} M^{(B)}_{S_n}$ now follows from a generalization of Pitt's theorem due to Nguyen[11] (see also Akcoglu and Krengel[12]). By the regularity the limit is again independent of the sequence chosen. This proves the existence with probability 1 of $q^{(B)}X^{(B)}$ and hence of $X^{(B)}$ (it is easy to prove that $q^{(B)} > 0$ with probability 1 when B is nonempty). Also $X^{(B)}$ will in general be a stochastic variable.

Thus we have now firmly established Eq. (3.40). As we have seen, except for the translation invariance only the regularity of the sequence $(L_n)_{n \geqslant 0}$ is required. This is a very weak condition (which, incidentally, may still be slightly relaxed[12]) and it is, of course, reassuring that we could have chosen instead of our L_n any other regular sequence of sets without affecting $q^{(B)}$ and $X^{(B)}$ [and Eq. (3.39a)]. Our choice of the cubes L_n is standard.

Equation (3.40) is the formal generalization of Eq. (3.10). The stochastic variables $q^{(B)}$ and $X^{(B)}$ are formally defined as limits, $q^{(B)}$ being the "asymptotic" density of black points corresponding to B and $X^{(B)}$ the "asymptotic" mean probability of a jump between two black points. In general, these limits will *not* be constant on $\mathscr{B}$, not even with probability 1. The simplest example for this situation is a color distribution which is a convex linear combination of two periodic color distributions with different densities of black points. We know only that

$$\overline{q^{(B)}}^{\mathscr{B}} = q \tag{3.41a}$$

$$\overline{q^{(B)}X^{(B)}}^{\mathscr{B}} = \overline{I[0 \in B] \sum_{l \in B} p(l)}^{\mathscr{B}} = q \operatorname{Prob}[n_1 = 1 \mid n_0 = 0] =: qX \tag{3.41b}$$

as may readily be shown.

In many cases of physical interest the limits $q^{(B)}$ and $X^{(B)}$ *are* constant with probability 1. These cases include all so-called *extremal* color distributions, which are distributions that cannot be written as a convex linear combination of two different (translation-invariant) color distributions. This follows from the easily established fact that $q^{(B)}$ and $X^{(B)}$ are translation invariant. Examples include all periodic distributions and all (translation-invariant) grand canonical distributions with "short-range correlations", i.e., having the property that the colorings of any two finite blocks become independent as the blocks are separated to infinity (see Ref. 13, Chap. 11). The random distribution falls in the latter class. In such cases Eq. (3.40) reduces to Eq. (3.10) through Eqs. (3.41a, b), so that again we end up with

$$\langle n_0 \rangle_0 \geqslant (1-q)^2/q(1-X) \tag{3.42}$$

To get this result we used $f_0 = 1$. When $f_0 < 1$ the generalization is, of course, given by Eq. (3.11).

In the random case obviously $X = p(0) + q[1 - p(0)]$ and we get the result $\langle n_0 \rangle_0 \geqslant (1-q)/q[1-p(0)]$. This is precisely the bound which for this case was known already from other arguments (see, e.g., Ref. 14) and happens to be valid also for asymmetric random walks. The fact that the bound obtained here is not stronger is not at all surprising. In Ref. 14 we proved that for the random case $\langle n_0 \rangle_0 \leqslant (1-q)/(1-F)q$, where F is the (total) probability of return to the origin. F can be arbitrarily close to $p(0)$ for lattices of large enough dimensionality, such that even within the class of symmetric random walks the lower bound obtained can be arbitrarily sharp.

(II) The Runs with Number $i \geqslant 1$. Our aim is to show that the inequalities in Eq. (3.31) remain valid in the general case. We shall not spell out the generalization in full detail but rather indicate the main parts of the proof.

First we introduce the analogs of Eqs. (3.19a, b) for given nonempty $B \in \mathscr{B}$:

$T_n^{(B)}(l \to l') :=$ probability for the walker, when starting from $l \in B$, to make a run of exactly n steps to $l' \in B$; $n \geqslant 1$; $l, l' \in B$.

$$T_{ll'}^{(B)} := \sum_n T_n^{(B)}(l \to l') \tag{3.43a}$$

$$S_{ll'}^{(B)} := \sum_n n T_n^{(B)}(l \to l') \tag{3.43b}$$

In the following we shall assume that $\mathscr{P}[B \text{ is empty}] = 0$. The probabilities $T_{ll'}^{(B)}$, $l, l' \in B$, form a matrix $\boldsymbol{T}^{(B)}$ of "transition" probabilities between the black points; $\boldsymbol{T}^{(B)}$ is (countably) infinite with probability 1. For given $l \in L$ let $\mathscr{B}_l := \{B \in \mathscr{B}: l \in B\}$. By the translation invariance and by Eq. (1.2)

$$\overline{\sum_{l' \in B} T_{ll'}^{(B)}}^{\mathscr{B}_l} = \overline{\sum_{l' \in B} T_{0l'}^{(B)}}^{\mathscr{B}_0} = 1 \qquad \text{for any given } l$$

Since obviously $\sum_{l' \in B} T_{ll'}^{(B)} \leqslant 1$ for all $l \in B$ and all $B \in \mathscr{B}$, this shows that

$$\sum_{l' \in B} T_{ll'}^{(B)} = 1 \qquad \text{for all } l \in B \text{ with probability } 1 \tag{3.44a}$$

A comparison of the random walk with its reversed counterpart shows that also

$$\sum_{l \in B} T_{ll'}^{(B)} = 1 \qquad \text{for all } l' \in B \text{ with probability 1} \tag{3.44b}$$

so that $\boldsymbol{T}^{(B)}$ is with probability 1 *doubly stochastic.*

Using Eqs. (3.43a, b) and Eq. (3.44a) we write out

$$\langle n_1 n_{k+1} \rangle_1 = \overline{\sum_{l_1, l_2, l_3 \in B} S_{0l_1}^{(B)} (\boldsymbol{T}^{(B)k-1})_{l_1 l_2} S_{l_2 l_3}^{(B)}}^{\mathscr{B}_0}, \qquad k \geqslant 1 \tag{3.45}$$

Next, using the translation invariance as well as the inversion symmetry of the lattice, we may write this in the following slightly different form:

$$\langle n_1 n_{k+1} \rangle_1 = \overline{\sum_{l_1, l_2, l_3 \in B} S_{l_1 0}^{(B)} (\boldsymbol{T}^{(B)k-1})_{0 l_2} S_{l_2 l_3}^{(B)}}^{\mathscr{B}_0} \tag{3.46}$$

The proof is left to the reader. Then, defining

$$S_l^{(B)} := \sum_{l' \in B} S_{ll'}^{(B)}, \qquad l \in B \tag{3.47}$$

and using the *symmetry* of the random walk, we get

$$\langle n_1 n_{k+1} \rangle_1 = \overline{\sum_{l \in B} S_0^{(B)} (\boldsymbol{T}^{(B)k-1})_{0l} S_l^{(B)}}^{\mathscr{B}_0} \tag{3.48}$$

Together with Eqs. (1.3) and (1.5) this gives

$$\langle n_k \rangle_0 = f_0^{-1} \overline{I[0 \in B] \sum_{l \in B} S_0^{(B)} (\boldsymbol{T}^{(B)k-1})_{0l} S_l^{(B)}}^{\mathscr{B}}, \qquad k \geqslant 1 \tag{3.49}$$

Equation (3.49) serves as the starting point for our calculation. In the following we shall *assume* that $\langle n_k \rangle_0 < \infty$ for all k. We shall return to this point in the discussion.

To arrive at our result we again use a truncation method. Consider Eq. (3.49). By the translation invariance the point 0 may be replaced by any given point l and if we then average over l in L_n we get

$$\langle n_k \rangle_0 = f_0^{-1} |L_n|^{-1} \overline{\sum_{l \in L_n \cap B} \sum_{l' \in L \cap B} S_l^{(B)} (\boldsymbol{T}^{(B)k-1})_{ll'} S_{l'}^{(B)}}^{\mathscr{B}} \qquad \text{for any } n \tag{3.50}$$

The second sum runs over the black points in the *whole* lattice. To obtain a symmetric expression we first restrict this sum to L_n and then take the limit

$n \to \infty$. Let $T_n^{(B)}$ be the *truncation* of $T^{(B)}$ obtained by deleting all rows and columns that correspond to black points *outside* L_n. Then

$$\langle n_k \rangle_0 = f_0^{-1} \lim_{n \to \infty} |L_n|^{-1} \overline{\sum_{l,l' \in L_n \cap B} S_l^{(B)} (T_n^{(B)k-1})_{ll'} S_{l'}^{(B)}}^{\mathscr{B}} \tag{3.51}$$

The equality sign is guaranteed by the fact that $L_n \to L$ monotonically in n.

We proceed by defining the following counterparts of Eqs. (3.13)–(3.14):

$$\Delta_k := \langle n_k \rangle_0 - \kappa, \qquad k \geq 1 \tag{3.52}$$

$$\gamma_k := \langle n_1 n_{k+1} \rangle_1 - \langle n_1 \rangle_1 \kappa, \qquad k \geq 1 \tag{3.53}$$

where

$$\kappa := f_0^{-1} \lim_{n \to \infty} |L_n|^{-1} \overline{|L_n \cap B|^{-1} \sum_{l,l' \in L_n \cap B} S_l^{(B)} S_{l'}^{(B)}}^{\mathscr{B}} \tag{3.54}$$

Equations (3.52)–(3.54) differ from Eqs. (3.13)–(3.14) for reasons which will become clear later. It will be seen that in the periodic case $\kappa = f_0 q^{-1} = \langle n_k \rangle_1$, so that the two sets of definitions coincide. Note that Eq. (3.15) remains valid and thus we can again investigate Δ_k by looking at γ_k.

Using Eqs. (3.15), (3.51), and (3.54) we write

$$\gamma_k = q^{-1} \lim_{n \to \infty} |L_n|^{-1} \overline{(s_n^{(B)}, (T_n^{(B)k-1} - |L_n \cap B|^{-1} E_n^{(B)}) s_n^{(B)})}^{\mathscr{B}} \tag{3.55}$$

where $E_n^{(B)}$ is the matrix with all elements equal to 1 of the same order as $T_n^{(B)}$, i.e., of order $|L_n \cap B|$, and $s_n^{(B)}$ is the vector with components $S_l^{(B)}$, $l \in L_n \cap B$. Now Eq. (3.55) is in form very similar to Eq. (3.29) and we shall use this fact to show that the γ_k satisfy *exactly the same* set of inequalities that were found in the periodic case. To that end let us write

$$\gamma_k = \overline{\gamma_k^{(B)}}^{\mathscr{B}} \tag{3.56}$$

with

$$\gamma_k^{(B)} := \lim_{n \to \infty} \gamma_{k;n}^{(B)} \tag{3.57a}$$

$$\gamma_{k;n}^{(B)} := q^{-1} |L_n|^{-1} (s_n^{(B)}, (T_n^{(B)k-1} - |L_n \cap B|^{-1} E_n^{(B)}) s_n^{(B)}) \tag{3.57b}$$

Here it is important that the limit in Eq. (3.57a) exists with probability 1. This is a consequence of the ergodic theorems used below Eq. (3.40) (note that $\gamma_k < \infty$, because $\langle n_k \rangle_0 < \infty$ by assumption), both the translation invariance and the regularity of our sequence $(L_n)_{n \geqslant 0}$ playing again an essential role. We shall show that with probability 1 the following inequalities hold:

$$\gamma_1^{(B)} \geqslant \gamma_3^{(B)} \geqslant \gamma_5^{(B)} \geqslant \gamma_7^{(B)} \geqslant \cdots \geqslant 0$$
$$\gamma_1^{(B)} \geqslant |\gamma_2^{(B)}|,\ \gamma_3^{(B)} \geqslant |\gamma_4^{(B)}|,\ \gamma_5^{(B)} \geqslant |\gamma_6^{(B)}|, \ldots \tag{3.58}$$

Together with Eqs. (3.15) and (3.56) this will immediately yield the desired generalization of Eq. (3.31).

Proof. As announced, to prove Eq. (3.58) we shall exploit the close resemblance between Eqs. (3.29) and (3.57b). Now for all B and n the matrix $\boldsymbol{T}_n^{(B)}$ is finite and, by the symmetry of the random walk, symmetric. Therefore we can imitate most of the argument that led from Eq. (3.29) to (3.30). The only difference is that, unlike $\boldsymbol{T}^{(B)}$, $\boldsymbol{T}_n^{(B)}$ is *not* doubly stochastic, as is required to follow the argument. According to Eqs. (3.44a, b) the doubly stochastic property is recovered in the limit as $n \to \infty$ and we have somehow to use this fact in Eqs. (3.57a, b). This is a technical problem which may be solved as follows.

Let $\boldsymbol{T}_n'^{(B)}$ be the matrix obtained from $\boldsymbol{T}_n^{(B)}$ by defining

$$(\boldsymbol{T}_n'^{(B)})_{ll'} := \begin{cases} (\boldsymbol{T}_n^{(B)})_{ll'}, & l' \neq l \\ 1 - \sum\limits_{l'' \neq l} (\boldsymbol{T}_n^{(B)})_{ll''}, & l' = l \end{cases} \tag{3.59}$$

i.e., by simply adding the "missing part" of the row sum (=column sum) to the diagonal elements. Like $\boldsymbol{T}_n^{(B)}$ this matrix is finite and symmetric, but by construction it is *also* doubly stochastic for *all n*. The point in introducing this matrix is that in the limit as $n \to \infty$ we may, as we shall show in a moment, simply replace $\boldsymbol{T}_n^{(B)}$ by $\boldsymbol{T}_n'^{(B)}$ in Eq. (3.57b) *without* affecting $\gamma_k^{(B)}$. But then $\gamma_{k;n}^{(B)}$ can be written in a diagonalized form which is similar to Eq. (3.30), $\boldsymbol{T}_n'^{(B)}$ having all the properties required to copy the proof, and Eq. (3.58) can immediately be read off. The details are left to the reader.

It thus only remains to show that the substitution of $\boldsymbol{T}_n'^{(B)}$ for $\boldsymbol{T}_n^{(B)}$ is indeed justified, in other words, that

$$d_k := \lim_{n \to \infty} |L_n|^{-1} \overline{(\boldsymbol{s}_n^{(B)}, [\boldsymbol{T}_n'^{(B)k-1} - \boldsymbol{T}_n^{(B)k-1}]\, \boldsymbol{s}_n^{(B)})}^{\mathscr{B}} = 0 \qquad \text{for all } k \geqslant 1 \tag{3.60}$$

This is done as follows. For $k=1$ there is nothing to prove, as then the term between the square brackets in Eq. (3.60) is zero. For $k=2$ we use Eqs. (3.44a) and (3.59) to write

$$d_2 = \lim_{n\to\infty} |L_n|^{-1} \overline{\sum_{l\in L_n\cap B} S_l^{(B)2} \left\{ \sum_{l'\in(L\setminus L_n)\cap B} T_{ll'}^{(B)} \right\}}^{\mathscr{B}}$$

The following reasoning is similar in spirit to the one used to prove Eq. (3.39a). Let

$$\varepsilon_{l;m}^{(B)} := \sum_{l'\in B,\,|l'-l|\geqslant m} T_{ll'}^{(B)}, \qquad l\in B$$

then for fixed m

$$d_2 \leqslant \lim_{n\to\infty} |L_n|^{-1} \overline{\sum_{l\in L_{n-m}\cap B} S_l^{(B)2} \varepsilon_{l;m}^{(B)}}^{\mathscr{B}} + \lim_{n\to\infty} |L_n|^{-1} \overline{\sum_{l\in L_{m,n}\cap B} S_l^{(B)2}}^{\mathscr{B}}$$

By the translation invariance it follows that for any $\varepsilon>0$

$$d_2 \leqslant q\, \overline{S_0^{(B)2}(\varepsilon_{0;m}^{(B)}+\varepsilon)}^{\mathscr{B}_0} \qquad \text{for all } m$$

When we now let $m\to\infty$ and observe that by Eq. (3.44a)

$$\lim_{m\to\infty} \varepsilon_{0;m}^{(B)} = 0 \qquad \text{with probability 1 in } \mathscr{B}_0$$

we find that

$$d_2 \leqslant \varepsilon q\, \overline{S_0^{(B)2}}^{\mathscr{B}_0} = \varepsilon q \langle n_1 n_2\rangle_1 = \varepsilon f_0 \langle n_1\rangle_0$$

[see Eq. (3.48)]. Here we have applied the *Lebesgue bounded convergence theorem* (see Ref. 15, p. 110) to interchange the limit and the average, using the fact that the average in the right-hand side is finite by assumption. Since ε is arbitrary this proves that $d_2=0$.

The proof for $k\geqslant 3$ proceeds in a similar way. We first note that $\boldsymbol{T}_n'^{(B)} \leqslant \boldsymbol{T}_n^{(B)} + \mathbf{1}_n^{(B)}$ and write

$$\begin{aligned}
\boldsymbol{T}_n'^{(B)k} - \boldsymbol{T}_n^{(B)k} &= \sum_{k'=0}^{k-1} \boldsymbol{T}_n'^{(B)k'} (\boldsymbol{T}_n'^{(B)} - \boldsymbol{T}_n^{(B)})\, \boldsymbol{T}_n^{(B)k-k'-1} \\
&\leqslant \sum_{k'=0}^{k-1} \sum_{k''=0}^{k'} \binom{k'}{k''} \boldsymbol{T}_n^{(B)k''} (\boldsymbol{T}_n'^{(B)} - \boldsymbol{T}_n^{(B)})\, \boldsymbol{T}_n^{(B)k-k'-1}
\end{aligned}$$

We substitute this inequality into Eq. (3.60) and follow the same type of reasoning as before, using the translation invariance. After a little manipulation we then find that

$$d_{k+1} \leqslant \varepsilon q \sum_{k'=0}^{k-1} \sum_{k''=0}^{k'} \binom{k'}{k''} \langle n_1 n_{k-k'+k''+1} \rangle_1$$
$$= \varepsilon f_0 \sum_{m=0}^{k-1} \binom{k}{m} \langle n_{m+1} \rangle_0$$

The details are left to the reader. The boundedness of the moments implies that $d_k = 0$ for all k. This proves Eq. (3.60) and hence Eq. (3.58). ∎

We have thus established the desired generalization of Eq. (3.31), except that it remains to identify κ in Eq. (3.54). Let

$$Y_n^{(B)} := |L_n \cap B|^{-1} \sum_{l \in L_n \cap B} S_l^{(B)} \tag{3.61}$$

then κ takes the form

$$\kappa = f_0^{-1} \lim_{n \to \infty} \overline{q_n^{(B)} Y_n^{(B)2}}^{\mathscr{B}}$$

where Eq. (3.37a) is used. We already showed that the limit in Eq. (3.39b) exists with probability 1. The same ergodic theorems imply that also

$$Y^{(B)} := \lim_{n \to \infty} Y_n^{(B)} \tag{3.62}$$

exists with probability 1 (note that $q^{(B)} > 0$ with probability 1), so that we arrive at

$$\kappa = f_0^{-1} \overline{q^{(B)} Y^{(B)2}}^{\mathscr{B}} \tag{3.63}$$

Like $q^{(B)}$, $Y^{(B)}$ will in general be a stochastic variable. A simple calculation shows that

$$\overline{q^{(B)} Y^{(B)}}^{\mathscr{B}} = q\, \overline{S_0^{(B)}}^{\mathscr{B}_0} = q \langle n_1 \rangle_1 = f_0 \tag{3.64}$$

Since $q^{(B)} \geqslant 0$ for all B we have

$$\overline{q^{(B)} Y^{(B)2}}^{\mathscr{B}} \cdot \overline{q^{(B)}}^{\mathscr{B}} - \{\overline{q^{(B)} Y^{(B)}}^{\mathscr{B}}\}^2 \geqslant 0$$

and therefore

$$\kappa \geqslant f_0 q^{-1} \tag{3.65}$$

Thus our earlier bounds remain valid in the general case. For *extremal* color distributions $q^{(B)}$ and $Y^{(B)}$ are constant with probability 1 and the equality sign holds in Eq. (3.65).

The above calculation completes the generalization of the bounds obtained in Section 3.1. What seems harder than in the periodic case, though, is to determine under what conditions $\Delta_k \to 0$ as $k \to \infty$ (it is not even obvious that $\Delta_k < \infty$) and, if so, how fast. In the periodic case we found that the decay, if present, is *always exponential* in k. In the general case we expect this *not* to be so. The decay depends on the eigenvalues of $T^{(B)}$ in the neighborhood of 1 and -1, the spectrum of $T^{(B)}$ can have both a discrete and a continuous part and the decay, if present, may generally be *slower* than exponential. What also seems harder than in the periodic case is to find examples where the equality signs hold in Eqs. (3.31) and (3.42). For the first run, however, we note the following. Returning to Eq. (3.49) we have

$$\langle n_1 \rangle_0 = \overline{S_0^{(B)2}}^{\mathscr{B}_0} / \overline{S_0^{(B)}}^{\mathscr{B}_0} \tag{3.66}$$

which is the counterpart of Eq. (3.26) (note that $\langle n_1 \rangle_0 - f_0 q^{-1}$ can be written as a variance), and it follows that $\langle n_1 \rangle_0 = f_0 q^{-1}$ *if and only if* within the set of color configurations that include the origin the average length of the first run is constant with probability 1. This may be used to construct new examples of equality for the first run, but it will be clear that the restriction on the color distribution and the random walk is rather strong, showing that in general there will be strict inequality.

4. AN APPLICATION TO LATTICES WITH TRAPS

Suppose now that the black points are *traps* characterized by a probability of escape η, i.e., whenever the walker visits a black point there is a probability $1-\eta$ that he is trapped (forever) and a probability η that he remains free ("escapes"). If $\eta = 0$ ($\eta > 0$) the trap is called *perfect (imperfect)*. We assume that $\eta < 1$.

Let

$T_n :=$ probability that the walker is trapped after exactly n steps; $n \geqslant 0$.

In Ref. 1 it is shown that T_n is monotone nonincreasing in n for *arbitrary* L, $\mathscr{P}$, and p. From Eq. (1.1) it may further be deduced that for all η the total probability of trapping $f := \sum_n T_n$ equals f_0. Thus by Eq. (2.2) $f = 1$

in almost all cases of physical interest. The average number of steps before trapping $\langle n \rangle := \sum_n nT_n / \sum_n T_n$ follows from

$$\langle n \rangle = \langle n_0 \rangle_0 + \sum_{i \geq 1} \eta^i \langle n_i \rangle_0 \tag{4.1}$$

Equation (4.1) shows that in order to calculate $\langle n \rangle$ one has to know all the moments $\langle n_i \rangle_0$, $i \geq 0$.

Our results in Section 3 give a lower bound for $\langle n \rangle$ for symmetric random walks. For perfect traps we have the bound given by Eq. (3.42) (assuming without loss of generality that $f = 1$ and that the trap distribution is extremal):

$$\langle n \rangle |_{\eta = 0} \geq (1-q)^2 / q(1-X) \tag{4.2a}$$

From Eq. (3.31), using that $\sum_{i=1}^k \langle n_i \rangle_0 \geq kq^{-1}$ for all k, we can further deduce that for imperfect traps $\langle n \rangle$ is prolonged by an amount

$$\langle n \rangle - \langle n \rangle |_{\eta = 0} \geq \frac{\eta}{1-\eta} \frac{1}{q} \tag{4.2b}$$

Except for a few rather special cases, the equality sign in Eq. (4.2b) holds only for strictly and pair-periodic trap distributions.

5. DISCUSSION

In Ref. 1 and the present paper we have studied statistical properties of the sequence of consecutive colors encountered by a random walker on a lattice of which the points are colored black and white according to a translation-invariant joint probability distribution. The relevance of our results to trapping problems in particular will be evident. Trapping problems have a long history and many properties have, in some form or other, been discussed in the literature (see, e.g., Refs. 3, 16, and 17). Usually, however, the traps are assumed to be distributed either periodically or randomly over the lattice. Except in one dimension, little is known in detail for other trap distributions. Reference 1 and the present paper are an attempt to bring out some of the characteristic features of trapping problems in a more general setting.

Earlier Results for Traps. For periodic trap distributions an *exact* solution for $\langle n \rangle$ (see Section 4) was found in Ref. 18. The approach followed in that paper is a generalization of earlier work by Montroll,[19] who derived an expression for $\langle n \rangle |_{\eta = 0}$. The final result, however, appears in a form that is not very practical for analytical purposes unless the unit cell of the periodicity contains a very limited number of traps. To be more specific, when N is the number of points in the unit cell and t the number of traps, $\langle n \rangle$ is expressed in terms of $t \times t$ determinants of which the elements are Green's functions that are N-fold sums depending on the positions of the traps and on the random walk. As an illustration of the difficulties that one encounters in this context the fact may serve that we have been unable to rederive Eqs. (4.2a, b) for any unit cell with more than two traps starting from the results of Refs. 18 and 19. Thus in practice to get detailed results one should have recourse to the computer.

For the random distribution asymptotic expansions for $\langle n \rangle$ valid for small q were obtained in Ref. 20 for several classes of random walks of varying dimensionality. Earlier work by Rosenstock[21] included a study of $\langle n \rangle |_{\eta=0}$ for $q \to 0$. So far, only few rigorous results have been obtained for the random case, except in one dimension. On the other hand, several approximative methods have been developed, one more sophisticated than the other, all for values of q that are either small or close to unity (see, e.g., Refs. 16 and 17).

Higher Runs, Odd/Even Effect. In this paper we have centered interest on the probabilities f_i and the moments $\langle n_i \rangle_0$. A particularly striking aspect of our results for the runs with number $i \geqslant 1$ is that for i odd $\langle n_i \rangle_0$ is always monotone in i whereas for i even a variety in behavior is displayed depending on the choice of $\mathscr{P}$ and p. This difference must essentially come from our assumption of symmetry, but it is not intuitively obvious why then the odd-numbered runs should be so special. To get some feeling for the situation, let us look at the first few runs in some more detail. For the sake of the argument we consider a *simple* random walk on a large unit cell, with periodic boundary conditions, in which the black points occur in several *large* compact "clusters" surrounded by a large "sea" of white points. Now if all black points would have equal probability to be the starting point for the first run we would have $\langle n_1 \rangle_0 = \langle n_1 \rangle_1 = q^{-1}$ by Eq. (1.3), but obviously the probability in question differs for different black points: by the translation invariance black points that are on the edge of a cluster are much more likely to be hit first than others. On the other hand, by the symmetry the average length of a run starting from the edge of a cluster is larger than that of one starting from the interior. Together this leads to $\langle n_1 \rangle_0 > q^{-1}$. The second black point hit is one that with a large probability lies either on the edge of a cluster or one layer deeper. This gives $q^{-1} < \langle n_2 \rangle_0 < \langle n_1 \rangle_0$. As more and more black points are visited, the "excess" in probability of points on the outer part of a cluster to be the next black point hit gradually "diffuses" into the cluster, so that $\langle n_i \rangle_0 \to q^{-1}$ *monotonically* as $i \to \infty$. When, instead of large clusters, we have clusters of *small size*, say, only one interior point and one boundary layer, then the following happens: after the first run the excess in probability of points on the edge is all transferred to the single interior point, so that now we get $\langle n_2 \rangle_0 < q^{-1}$. By the symmetry this effect will be reversed by an extra run, so that again $\langle n_3 \rangle_0 > q^{-1}$ etc. and the decay is in this case *oscillating*.

To illustrate this argument one may consider a simple random walk on a ring of arbitrary length with one compact cluster of black points of size M. An easy calculation shows that as M varies one gets the following types of behavior: (a) $M = 1, 2$: $\langle n_i \rangle_0 = q^{-1}$ for all $i \geqslant 1$; (b) $M = 3$: oscillating decay; (c) $M = 4$: $\langle n_1 \rangle_0 > q^{-1}$, equality for all other i; (d) $M \geqslant 5$: monotonic decay.

The extreme example considered above brings out the origin of our results for the runs with number $i \geqslant 1$. More generally it will be clear that the bounds obtained all arise from the fact that (i) *different black points may have different environments*, (ii) black points in certain environments are *more easily accessible* than others, i.e., are favoured over others to be hit at a given stage in the process, (iii) those black points which are most easily accessible are *also* the ones from which a run takes longest, simply because they are surrounded by more white points. The "clumping" of black points, or to phrase it differently, the spatial fluctuation in local ordering of colors generally tends to increase the lengths of the runs and to favor monotonic behavior, but certain special conditions may cause a shortening of the even-numbered runs. It is interesting to note that the first run in a sense "sets the stage" for all the subsequent runs.

The bounds obtained for the runs with number $i \geqslant 1$ are probably fairly strong. In Eqs. (3.31) and (4.2b) the equality signs hold for *all* strictly and pair-periodic distributions, regardless of the random walk and the size of the unit cell, and so there is every reason to expect that the inequalities will be sharp in many other cases of physical interest. Moreover, in Ref. 20 it is shown that for the random distribution $\langle n \rangle - \langle n \rangle|_{\eta=0} \simeq \eta/(1-\eta)\, q$ as $q \to 0$ for a very large class of random walks, including all transient random walks (which include all aperiodic random walks with $d \geqslant 3$). Since η is arbitrary this implies that $\langle n_i \rangle_0 \simeq 1/q$, $q \to 0$, for all $i \geqslant 1$, so that equality in Eqs. (3.31) and (4.2b) holds asymptotically in this case.

Zeroth Run. The zeroth run differs in character from all the subsequent runs and the bound obtained in Eq. (3.42) is in practice, unfortunately, not so strong. For strictly periodic distributions, for instance, it can be shown that the equality sign holds *only* for a very special type of random walk, which we have called "indifferent" with respect to the unit cell, and that for most other random walks the bound is numerically rather weak when the unit cell is large. Moreover, for the random distribution it is known that $\langle n_0 \rangle_0 \simeq 1/(1-F)\, q$, $q \to 0$, for transient random walks, where F is the (total) probability of return to the origin (in the absence of traps), and thus for small q the bound is in this case "off" by a factor $1/(1-F)$.

Our result for the zeroth run becomes more transparent when we rewrite Eq. (3.42) as an inequality relating two conditional averages:

$$\langle n_0 \,|\mathrm{W} \rangle \geqslant \langle n_1 - 1 \,|\mathrm{BW} \rangle \tag{5.1}$$

Here Eq. (1.3) is used and W (BW) is a condition on the first (two) color(s) encountered by the walker. Returning to our example of a simple random walk on a unit cell with clusters of black points, we see that Eq. (5.1) is quite clear: when the walker may start anywhere in the sea of white points it takes him longer to reach a black point than when he must first step on a black point, next step to a white point and *then* start to go for a black point, simply because in the latter case he starts next to a cluster. In the general case Eq. (5.1) comes from the fact that (i) different *white* points may have different environments, (ii) white points in certain environments are *more easily hit* from a black point than others, (iii) those white points which are most easily hit are *also* the ones from which a run takes shortest. This is very similar to what we listed earlier with respect to the behavior of the other runs. It now also becomes clear why our bound is not so strong when q is small: in the case of a strictly periodic distribution, for instance, black points *do* but white points generally *do not* have the same environment, and in particular when the unit cell is large (and hence $\langle n_0 \rangle_0$ large) this will have a substantial effect. Equation (5.1) tends to become better as q increases. Thus it remains a challenge for the zeroth run in particular to look for ways of obtaining a better bound. It is amusing to note that Eq. (5.1) reduces to an equality for all color distributions which are complementary to a strictly or pair-periodic distribution, i.e., obtained from the latter ones after changing black into white and vice versa. Note also that Eq. (3.42) is the only inequality obtained that depends explicitly on $\mathscr{P}$ and p.

Finiteness of Moments. A question which we have postponed so far is whether or not the averages that we consider are finite [see below Eqs. (3.33) and (3.49)]. For periodic distributions all moments are finite because of the finite size of the unit cell. For general distributions, however, Eq. (1.3) is exceptional in that $\langle n_1 \rangle_1$ is the only moment that is always finite, and there are examples where *all* other moments are infinite. In such cases the bounds obtained are, of course, trivial though still correct. (It is not hard to show that the moments $\langle n_i \rangle_0$, $i \geqslant 1$, are either all finite or all infinite.) As an example, take a simple random walk on $\mathbb{Z}$ and let the color distribution be such that, loosely described, the lengths of white intervals between black points are independently and identically distributed. Let $C(m)$ denote the probability that an interval has length m. We have $\sum_m C(m) = 1$ and $\sum_m mC(m) = q^{-1}$, but C is otherwise arbitrary. Following the approach of Ref. 19 it may then be shown that $\langle n_0 \rangle_0 = \sum_m (m^3 - m)\, C(m)/6\sum_m mC(m)$, which can be made infinite by choosing C such that $\sum_m m^3 C(m) = \infty$. It may further be shown that when C is such that $\sum_m m^2 C(m) = \infty$, also all the next runs have infinite first moment. This example is, of course, highly special and it seems reasonable to expect that in most cases of physical interest the moments considered are finite. In particular for the random distribution one should be able to establish finiteness for arbitrary random walk.

Assumptions Used. The assumption of symmetry of the random walk plays a crucial role in most of the present paper. Once the symmetry is dropped one may get "wildly varying" results [see the remarks below Eq. (3.1a)] and each of the inequalities obtained may be seriously violated. To illustrate this let us consider a pair-periodic color distribution. It follows from Eqs. (3.20a, b) that $T_{11} = T_{22} = 1 - T_{12} = 1 - T_{21}$. Since there are only two traps in the unit cell it further follows that $S_{11} = S_{22}$ (!) [see below Eq. (3.27)], and with Eqs. (3.13)–(3.15) and (3.28) this gives $\langle n_i \rangle_0 = q^{-1} - \frac{1}{4} q (S_{12} - S_{21})^2 (T_{11} - T_{12})^{i-1}$, $i \geqslant 1$ (assume $f_0 = 1$). Now for symmetric random walks $S_{12} = S_{21}$ and $\langle n_i \rangle_0 = q^{-1}$, as found earlier. For asymmetric random walks, however, we have in general $S_{12} \neq S_{21}$, so that $\langle n_1 \rangle_0 < q^{-1}$ and when $T_{11} \neq T_{12}$ also $\langle n_i \rangle_0 < q^{-1}$ for all i odd. This is just the *opposite* of what we found in Eq. (3.31). Thus, for our bounds the symmetry is necessary.

By the translation invariance of $\mathscr{P}$ and p the sequence of colors encountered by the walker is a *stationary* stochastic process. We have derived the results in Ref. 1 on the basis of this property *alone*, without referring to the detailed background of the process, i.e., without using the fact that the color sequence is actually constructed from a random walk taking place on a stochastically colored lattice. Therefore Eqs. (1.1)–(1.7) reflect only this stationarity and, viewed in retrospect, they could also have been derived starting from certain theorems on stationary stochastic processes known from the mathematical literature. The reader is referred to Breiman[22] (Chap. 6) and Berbee[23] (Chap. 3).

For the present paper the situation is quite different: in deriving our results we have made frequent use of properties of the underlying model such as lattice structure, existence of asymptotic density of black points, independence of successive steps of the walker, symmetry of steps and paths, etc. Therefore the various inequalities obtained in the present paper reflect more of the specific features of our model.

To conclude the discussion it seems appropriate to ask: "How restrictive are the basic assumptions in our model in view of actual applications?,, The model may be used to describe physical processes such as the diffusion and trapping of "particles" in a medium with static traps. In such processes the translation invariance enters as a very natural assumption: the system, though microscopically inhomogeneous, is assumed to be *statistically homogeneous*, and hence homogeneous on a macroscopic level. As to the symmetry, this assumption should be realistic for a system *without external field*, where the stepping probability distribution of the "particles" is expected to exhibit the symmetries of the underlying lattice structure.

ACKNOWLEDGMENT

Part of this work was done with the financial support of the "Stichting voor Fundamenteel Onderzoek der Materie" (FOM). We are grateful to Dr. H. C. P. Berbee for calling our attention to Refs. 22 and 23 and we wish to thank him for several interesting discussions.

REFERENCES

1. W. Th. F. den Hollander and P. W. Kasteleyn, *Physica* **117A**:179 (1983).
2. W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. 1, 3rd ed. (Wiley, New York, 1968) and Vol. 2, 2nd ed. (Wiley, New York, 1971).
3. F. Spitzer, *Principles of Random Walk*, 2nd ed. (Springer, New York, 1976).
4. E. Seneta, *Non-negative Matrices and Markov Chains*, 2nd ed. (Springer, New York, 1980).
5. D. R. Cox and H. D. Miller, *The Theory of Stochastic Processes* (Methuen, London, 1965).
6. M. Marcus and H. Minc, *A Survey of Matrix Theory and Matrix Inequalities* (Allyn and Bacon, Boston, 1964).
7. E. F. Beckenbach and R. Bellman, *Inequalities* (Springer, Berlin, 1961).
8. E. W. Montroll, *Proc. Symp. Appl. Math.* **16**:193 (1964).
9. M. Parodi, *La Localisation des Valeurs Caractéristiques des Matrices et ses Applications* (Gauthiers-Villars, Paris, 1959).
10. H. R. Pitt, *Proc. Cambridge Philos. Soc.* **38**:325 (1942).
11. X. X. Nguyen, *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **48**:159 (1979).
12. M. A. Akcoglu and U. Krengel, *J. Reine Angew. Math.* **323**:53 (1981).
13. C. J. Preston, *Gibbs States on Countable Sets, Cambridge Tracts in Mathematics* **68** (Cambridge University Press, Cambridge, 1974).
14. P. W. Kasteleyn and W. Th. F. den Hollander, *J. Stat. Phys.* **30**:363 (1983).
15. P. R. Halmos, *Measure Theory* (Springer, New York, 1974).
16. G. H. Weiss and R. J. Rubin, *Adv. Chem. Phys.* **52**:363 (1982).
17. Proceedings of a Symposium on Random Walks, U.S. National Bureau of Standards, Gaithersburg, Maryland, 1982, in *J. Stat. Phys.*, **30**:(2) (1983).
18. W. Th. F. den Hollander and P. W. Kasteleyn, *Physica* **112A**:523 (1982).

19. E. W. Montroll, *J. Phys. Soc. Japan Suppl.* **26**:6 (1969).
20. W. Th. F. den Hollander, *J. Stat. Phys.* **37**:331 (1984).
21. H. B. Rosenstock, *J. Math. Phys.* **21**:1643 (1980).
22. L. Breiman, *Probability* (Addison-Wesley, Reading, Massachusetts, 1968).
23. H. C. P. Berbee, Random walks with stationary increments and renewal theory (thesis), *Mathematical Centre Tracts* **112** (Amsterdam, 1979).